1. Let $X_1, \ldots, X_n$ be i.i.d. from the Gamma distribution $\Gamma(\alpha, \gamma)$ with unknown $\alpha$ and $\gamma$, whose p.d.f. is

$$
 f(x) = \frac{1}{\Gamma(\alpha) \gamma^\alpha} x^{\alpha-1} e^{-x/\gamma} \mathbb{1}(x > 0).
$$

(i) Show that $\Gamma(\alpha, \gamma)$ belongs to an exponential family.

(ii) Find a sufficient statistic for $(\alpha, \gamma)$.

2. Let $X_1, \ldots, X_n$ be i.i.d. from $\Gamma(\alpha, \gamma)$.

(i) For testing $H_0 : \alpha \leq \alpha_0$ versus $H_1 : \alpha > \alpha_0$, and $H_0 : \alpha = \alpha_0$ versus $H_1 : \alpha \neq \alpha_0$, show that there exist UMP unbiased tests whose rejections are based on $W = \prod_{i=1}^n \left( X_i / \bar{X} \right)$.

(ii) For testing $H_0 : \gamma \leq \gamma_0$ versus $H_1 : \gamma > \gamma_0$, show that a UMP unbiased test rejects $H_0$ when $
 \sum_{i=1}^n X_i > C(\prod_{i=1}^n X_i)$. Here, $C(t)$ is a function of $t$.

3. Let $X$ and $Y$ be independently distributed according to negative binomial distributions $\text{Nb}(p_1, m)$ and $\text{Nb}(p_2, n)$ respectively, and let $q_i = 1 - p_i$.

(i) There exists a UMP unbiased test for testing $H_0 : p_1 \leq p_2$ versus $H_0 : p_1 > p_2$.

(ii) Determine the conditional distribution required for testing $H_0$ when $m = n = 1$. 