Math 281C Homework 3 Solutions

Throughout the solutions, suppose $X_1, \dots, X_n$ are random variables with realized values $x_1, \dots, x_n$, we denote $\boldsymbol{X} = (X_1, \dots, X_n)$ and $\boldsymbol{x} = (x_1, \dots, x_n)$.

1. Let $X_1, \dots, X_n$ be i.i.d. from $N(\theta, \sigma^2)$ with σ^2 known. Consider testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$. Define the test that rejects H_0 if and only if

$$\bar{X} > \sigma z_{\alpha/2}/\sqrt{n} + \theta_0 \quad \text{or} \quad \bar{X} < -\sigma z_{\alpha/2}/\sqrt{n} + \theta_0,$$

where $z_{\alpha/2}$ is the upper $\alpha/2$-quantile of $N(0,1)$, and $\bar{X} = (1/n)\sum_{i=1}^n X_i$. Verify that this test a UMP unbiased (UMPU) level α test.

Solution: The joint density is

$$f(\boldsymbol{x}) = (2\pi\sigma^2)^{-n/2} \exp\left\{ \frac{n\theta\bar{x}}{\sigma^2} - \frac{n\theta^2}{2\sigma^2} \right\} \exp\left\{ -\frac{\sum_{i=1}^n x_i^2}{2\sigma^2} \right\}.$$

By Theorem 5.2.2, a UMPU test rejects H_0 if $\bar{x} < c_1$ or $\bar{x} > c_2$, and c_1, c_2 satisfy equations (2.6) and (2.7). In addition, by Remark 5.2.1 (2), it suffices to verify (2.6). To this end, notice that

$$\mathbb{P}(\bar{X} > \sigma z_{\alpha/2}/\sqrt{n} + \theta_0) = \mathbb{P}\{(\bar{X} - \theta_0)\sqrt{n}/\sigma > z_{\alpha/2}\} = \mathbb{P}(Z > z_{\alpha/2}) = \alpha/2,$$

where $Z \sim \mathcal{N}(0,1)$, and similarly, $\mathbb{P}(\bar{X} < -\sigma z_{\alpha/2}/\sqrt{n} + \theta_0) = \alpha/2$. This completes the verification of (2.6).

2. Let $X_1, \dots, X_{10}$ be i.i.d. from Bernoulli(p).

(i) Find a UMP test of size $\alpha = 0.1$ for testing $H_0 : p \leq 0.2$ or $p \geq 0.7$ versus $H_1 : 0.2 < p < 0.7$.

Solution: The joint density is

$$f(\boldsymbol{x}) = \exp\left\{ \log\frac{p}{1-p} \sum_{i=1}^n x_i + n\log(1-p) \right\}.$$

By Theorem 4.4.1, a UMP test of size α is

$$\phi(\boldsymbol{x}) = \begin{cases} 1 & \text{when } C_1 < T(\boldsymbol{x}) < C_2, \\ \gamma_i & \text{when } T(\boldsymbol{x}) = C_i, \quad i = 1, 2, \\ 0 & \text{otherwise,} \end{cases}$$

where $T(\boldsymbol{x}) = \sum_{i=1}^n x_i$, and $T(\boldsymbol{X}) = \sum_{i=1}^n X_i \sim \text{Binomial}(n, p)$. The four unknowns $C_1, C_2, \gamma_1, \gamma_2$ satisfy equation set (4.5). By enumerating integer pairs for (C_1, C_2) and solving (4.5), we have $C_1 = 4, C_2 = 5, \gamma_1 = 0.945, \gamma_2 = 0.634$. In other words, the UMP test of size α is

$$\phi(\boldsymbol{x}) = \begin{cases} 0.945 & \text{when } \sum_{i=1}^n x_i = 4, \\ 0.634 & \text{when } \sum_{i=1}^n x_i = 5, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) Find the power of the UMP test in (i) when $p = 0.4$.

Solution: The power is $\mathbb{E}_{p=0.4}[\phi(\boldsymbol{X})] = 0.364$.

(iii) Find a UMP unbiased test of size $\alpha = 0.1$ for testing $H_0 : p = 0.2$ versus $H_1 : p \neq 0.2$.

Solution: By Theorem 5.2.2, a UMPU test of size α is

$$\phi(\boldsymbol{x}) = \begin{cases} 1 & \text{when } T(\boldsymbol{x}) < C_1 \text{ or } T(\boldsymbol{x}) > C_2, \\ \gamma_i & \text{when } T(\boldsymbol{x}) = C_i, \quad i = 1,2, \\ 0 & \text{otherwise,} \end{cases}$$

and the four unknowns $C_1, C_2, \gamma_1, \gamma_2$ satisfy equations (2.6) and (2.7). Solving them similarly as in part (i), we obtain that $C_1 = 0, C_2 = 4, \gamma_1 = 0.559, \gamma_2 = 0.082$. Hence, the UMPU test of size α is

$$\phi(\boldsymbol{x}) = \begin{cases} 1 & \text{when } \sum_{i=1}^n x_i > 4, \\ 0.559 & \text{when } \sum_{i=1}^n x_i = 0, \\ 0.082 & \text{when } \sum_{i=1}^n x_i = 4, \\ 0 & \text{otherwise.} \end{cases}$$

(iv) Find the power of the UMP unbiased test in (iii) when $p = 0.4$.

Solution: The power is $\mathbb{E}_{p=0.4}[\phi(\boldsymbol{X})] = 0.391$.

3. Let $X_1, \dots, X_n$ be i.i.d. from some distribution function $F_\theta(x)$. Find a UMP unbiased test for testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$ if

(i) $F_\theta(\cdot)$ is the CDF of Poisson(θ), that is,

$$\mathbb{P}_\theta(X = x) = \theta^x e^{-\theta}/x!, \quad x = 0, 1, 2, \dots \quad \text{and} \quad \theta > 0.$$

In this case, $\theta_0 > 0$.

Solution: The solutions are similar to Question 2, part (iii). A UMPU test of size α is

$$\phi(\boldsymbol{x}) = \begin{cases} 1 & \text{when } T(\boldsymbol{x}) < C_1 \text{ or } T(\boldsymbol{x}) > C_2, \\ \gamma_i & \text{when } T(\boldsymbol{x}) = C_i, \quad i = 1,2, \\ 0 & \text{otherwise,} \end{cases}$$

where $T(\boldsymbol{X}) = \sum_{i=1}^n X_i \sim \text{Poisson}(n\theta)$, and the four unknowns $C_1, C_2, \gamma_1, \gamma_2$ satisfy equations (2.6) and (2.7).

(ii) $F_\theta(\cdot)$ is the CDF of Geometric(θ), that is,

$$\mathbb{P}_\theta(X = x) = (1-\theta)^{x-1}\theta, \quad x = 1, 2, \dots \quad \text{and} \quad 0 \leq \theta \leq 1.$$

In this case, $0 < \theta_0 < 1$.

Solution: A UMPU test of size α is

$$\phi(\boldsymbol{x}) = \begin{cases} 1 & \text{when } T(\boldsymbol{x}) < C_1 \text{ or } T(\boldsymbol{x}) > C_2, \\ \gamma_i & \text{when } T(\boldsymbol{x}) = C_i, \quad i = 1,2, \\ 0 & \text{otherwise,} \end{cases}$$

where $T(\boldsymbol{X}) = \sum_{i=1}^n X_i \sim$ Negative Binomial$(n\theta)$, and the four unknowns $C_1, C_2, \gamma_1, \gamma_2$ satisfy equations (2.6) and (2.7).

4. Let X and Y be independently distributed with Poisson distributions Poisson(λ) and Poisson(μ). Find the power of the UMP unbiased test of $H_0 : \mu \leq \lambda$ versus the alternative $\lambda = 1, \mu = 2$. at level of significance $\alpha = 0.1$.

Solution: The joint density is

$$f(x,y) = e^{-(\lambda+\mu)} \frac{\exp\{x \log\lambda + y \log\mu\}}{x!y!} = e^{-(\lambda+\mu)} \frac{\exp\{(x+y)\log\lambda + y\log(\mu/\lambda)\}}{x!y!}.$$

Denote $U = Y \sim \text{Poisson}(\mu)$ and $T = X + Y \sim \text{Poisson}(\lambda + \mu)$, and $\theta = \log(\mu/\lambda)$. Then, the original test can be equivalently converted to

$$H_0 : \theta \leq 0 \quad \text{versus} \quad H_1 : \theta = \log 2.$$

By Theorem 5.3.3 (1), A UMPU test of size α is

$$\phi(u,t) = \begin{cases} 1 & \text{when } u > C(t), \\ \gamma(t) & \text{when } u = C(t), \\ 0 & \text{otherwise,} \end{cases}$$

where $C(t), \gamma(t)$ satisfy $\mathbb{E}_{\theta=0}[\phi(U,T)|T=t] = \alpha$, and $[U|T=t,\theta=0] \sim \text{Binomial}(t, 1/2)$.
To calculate the power, notice that under H_1, $T \sim \text{Poisson}(3)$ and $[U|T=t] \sim \text{Binomial}(t, 2/3)$, and the power is $\mathbb{E}_{\lambda=1,\mu=2}[\phi(U,T)]$.