Math 281C Homework 3 Solutions

Throughout the solutions, suppose $X_1, \ldots, X_n$ are random variables with realized values $x_1, \ldots, x_n$, we denote $X = (X_1, \ldots, X_n)$ and $x = (x_1, \ldots, x_n)$.

1. Let $X_1, \ldots, X_n$ be i.i.d. from $N(\theta, \sigma^2)$ with $\sigma^2$ known. Consider testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$. Define the test that rejects $H_0$ if and only if

   $$\bar{X} > \sigma z_{\alpha/2}/\sqrt{n} + \theta_0 \quad \text{or} \quad \bar{X} < -\sigma z_{\alpha/2}/\sqrt{n} + \theta_0,$$

where $z_{\alpha/2}$ is the upper $\alpha/2$-quantile of $N(0, 1)$, and $\bar{X} = (1/n) \sum_{i=1}^n X_i$. Verify that this test a UMP unbiased (UMPU) level $\alpha$ test.

**Solution:** The joint density is

$$f(x) = (2\pi \sigma^2)^{-n/2} \exp \left\{ \frac{n \theta \bar{x} - \frac{n \theta^2}{2 \sigma^2}}{\sigma^2} \right\} \exp \left\{ -\frac{\sum_{i=1}^n x_i^2}{2 \sigma^2} \right\}.$$  

By Theorem 5.2.2, a UMPU test rejects $H_0$ if $\bar{x} < c_1$ or $\bar{x} > c_2$, and $c_1, c_2$ satisfy equations (2.6) and (2.7). In addition, by Remark 5.2.1 (2), it suffices to verify (2.6). To this end, notice that

$$\mathbb{P}(\bar{X} > \sigma z_{\alpha/2}/\sqrt{n} + \theta_0) = \mathbb{P}\left((\bar{X} - \theta_0)/\sigma > z_{\alpha/2}\right) = \mathbb{P}(Z > z_{\alpha/2}) = \alpha/2,$$

where $Z \sim N(0, 1)$, and similarly, $\mathbb{P}(\bar{X} < -\sigma z_{\alpha/2}/\sqrt{n} + \theta_0) = \alpha/2$. This completes the verification of (2.6).

2. Let $X_1, \ldots, X_{10}$ be i.i.d. from Bernoulli($p$).

   (i) Find a UMP test of size $\alpha = 0.1$ for testing $H_0 : p \leq 0.2$ or $p \geq 0.7$ versus $H_1 : 0.2 < p < 0.7$.

   **Solution:** The joint density is

   $$f(x) = \exp \left\{ \log \frac{p}{1-p} \sum_{i=1}^n x_i + n \log(1-p) \right\}.$$  

By Theorem 4.4.1, a UMP test of size $\alpha$ is

$$\phi(x) = \begin{cases} 1 & \text{when } C_1 < T(x) < C_2, \\ \gamma_i & \text{when } T(x) = C_i, \ i = 1, 2, \\ 0 & \text{otherwise}, \end{cases}$$

where $T(x) = \sum_{i=1}^n x_i$, and $T(X) = \sum_{i=1}^n X_i \sim \text{Binomial}(n, p)$. The four unknowns $C_1, C_2, \gamma_1, \gamma_2$ satisfy equation set (4.5). By enumerating integer pairs for $(C_1, C_2)$ and solving (4.5), we have $C_1 = 4, C_2 = 5, \gamma_1 = 0.945, \gamma_2 = 0.634$. In other words, the UMP test of size $\alpha$ is

$$\phi(x) = \begin{cases} 0.945 & \text{when } \sum_{i=1}^n x_i = 4, \\ 0.634 & \text{when } \sum_{i=1}^n x_i = 5, \\ 0 & \text{otherwise}. \end{cases}$$

(ii) Find the power of the UMP test in (i) when $p = 0.4$.

   **Solution:** The power is $\mathbb{E}_{p=0.4}[\phi(X)] = 0.364.$
3. Let $X_1, \ldots, X_n$ be i.i.d. from some distribution function $F_\theta(x)$. Find a UMP unbiased test for testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$ if

(i) $F_\theta(\cdot)$ is the CDF of Poisson($\theta$), that is,
\[ P_\theta(X = x) = \theta^x e^{-\theta} / x!, \quad x = 0, 1, 2, \ldots \quad \text{and} \quad \theta > 0. \]

In this case, $\theta_0 > 0$.

Solution: The solutions are similar to Question 2, part (iii). A UMPU test of size $\alpha$ is
\[ \phi(x) = \begin{cases} 
1 & \text{if } T(x) < C_1 \text{ or } T(x) > C_2, \\
\gamma_i & \text{if } T(x) = C_i, \quad i = 1, 2, \\
0 & \text{otherwise,}
\end{cases} \]

where $T(X) = \sum_{i=1}^{n} X_i \sim \text{Poisson}(n\theta)$, and the four unknowns $C_1, C_2, \gamma_1, \gamma_2$ satisfy equations (2.6) and (2.7).

(ii) $F_\theta(\cdot)$ is the CDF of Geometric($\theta$), that is,
\[ P_\theta(X = x) = (1 - \theta)^{x-1} \theta, \quad x = 1, 2, \ldots \quad \text{and} \quad 0 \leq \theta \leq 1. \]

In this case, $0 < \theta_0 < 1$.

Solution: A UMPU test of size $\alpha$ is
\[ \phi(x) = \begin{cases} 
1 & \text{if } T(x) < C_1 \text{ or } T(x) > C_2, \\
\gamma_i & \text{if } T(x) = C_i, \quad i = 1, 2, \\
0 & \text{otherwise,}
\end{cases} \]

where $T(X) = \sum_{i=1}^{n} X_i \sim \text{Negative Binomial}(n\theta)$, and the four unknowns $C_1, C_2, \gamma_1, \gamma_2$ satisfy equations (2.6) and (2.7).

4. Let $X$ and $Y$ be independently distributed with Poisson distributions Poisson($\lambda$) and Poisson($\mu$). Find the power of the UMP unbiased test of $H_0 : \mu \leq \lambda$ versus the alternative $\lambda = 1, \mu = 2$. at level of significance $\alpha = 0.1$.

Solution: The joint density is
\[ f(x, y) = e^{-(\lambda+\mu)} \frac{\exp\{x \log \lambda + y \log \mu\}}{x!y!} = e^{-(\lambda+\mu)} \frac{\exp\{(x+y) \log \lambda + y \log (\mu/\lambda)\}}{x!y!}. \]

Denote $U = Y \sim \text{Poisson}(\mu)$ and $T = X + Y \sim \text{Poisson}(\lambda + \mu)$, and $\theta = \log(\mu/\lambda)$. Then, the original test can be equivalently converted to
\[ H_0 : \theta \leq 0 \quad \text{versus} \quad H_1 : \theta = \log 2. \]
By Theorem 5.3.3 (1), a UMPU test of size $\alpha$ is

$$
\phi(u,t) = \begin{cases} 
1 & \text{when } u > C(t), \\
\gamma(t) & \text{when } u = C(t), \\
0 & \text{otherwise},
\end{cases}
$$

where $C(t), \gamma(t)$ satisfy $E_{\theta=0}[\phi(U,T)|T=t] = \alpha$, and $[U|T=t, \theta=0] \sim \text{Binomial}(t,1/2)$.

To calculate the power, notice that under $H_1$, $T \sim \text{Poisson}(3)$ and $[U|T=t] \sim \text{Binomial}(t,2/3)$, and the power is $E_{\lambda=1, \mu=2}[\phi(U,T)]$. 