Throughout the solutions, uppercase letters denote random variables, and lowercase letters denote realized values.

1. Suppose $X$ is one observation from a population with beta$(\theta, 1)$ pdf—$Cx^{\theta-1}$ for $0 < x < 1$.

(a) For testing $H_0: \theta \leq 1$ versus $H_1: \theta > 1$, find the size and sketch the power function of the test that rejects $H_0$ if $X > 1/2$.

**Solution:** It follows from $\int_1^1 Cx^{\theta-1}dx = 1$ that $C = \theta$. For any $\theta > 0$,

$$P(X > 1/2|\theta) = 1 - (1/2)^\theta,$$

so the size is $\sup_{\theta \geq 1} P(X > 1/2|\theta) = 1/2$. The power curve can be accordingly drawn.

(b) Find the most powerful level $\alpha$ test of $H_0: \theta = 1$ versus $H_1: \theta = 2$.

**Solution:** By the N-P Lemma, we reject $H_0$ if

$$\lambda = \frac{f(x|\theta = 2)}{f(x|\theta = 1)} = 2x > c.$$

The value $c$ satisfies $\alpha = P(2X > c|\theta = 1)$, which yields $c = 2(1 - \alpha)$.

(c) Is there a UMP test of $H_0: \theta \leq 1$ versus $H_1: \theta > 1$? If so, find it; if not, prove so.

**Solution:** Yes. It can be shown that the distribution has MLR in $x$, and the existence of UMP is guaranteed by Theorem 3.2.1. The test rejects $H_0$ if $x \geq 1 - \alpha$.

2. Let $X$ be one observation from a Cauchy scale distribution with density

$$f_\theta(x) = \frac{\theta}{\pi \theta^2 + x^2}, \quad -\infty < x < \infty, \theta > 0.$$

(a) Show that this family does not have an MLR in $x$.

**Solution:** For any $\theta_2 > \theta_1$,

$$\frac{f(x|\theta_2)}{f(x|\theta_1)} = \frac{\theta_2}{\theta_1} \frac{\theta_1^2 + x^2}{\theta_2^2 + x^2},$$

and its derivative depends on the sign of $x$, so the above function is not monotonic in $x$.

(b) Show that the distribution of $|X|$ does have an MLR.

**Solution:** By symmetry, the density of $|X|$ is

$$f_\theta(x) = \frac{2\theta}{\pi \theta^2 + x^2}, \quad 0 \leq x < \infty, \theta > 0.$$

The likelihood ratio can be similarly calculated, and is monotonic in $x$.

3. Let $X$ be one observation from a Cauchy distribution

$$f_\theta(x) = \frac{C}{1 + (x - \theta)^2}, \quad x \in \mathbb{R}.$$

(a) Show that this family does not have an MLR in $x$.

**Solution:** For any $\theta_2 > \theta_1$,

$$\lambda(x|\theta_2, \theta_1) := \frac{f(x|\theta_2)}{f(x|\theta_1)} = \frac{1 + (x - \theta_1)^2}{1 + (x - \theta_2)^2}.$$

We have $\lim_{x \to \infty} \lambda(x|\theta_2, \theta_1) = \lim_{x \to \infty} \lambda(x|\theta_2, \theta_1) = 1$, and $\lambda(0) \neq 1$, so $\lambda(x|\theta_2, \theta_1)$ is not monotonic in $x$. Alternatively, we can calculate its derivative to reach the same conclusion.
4. Let $X$ where $k$ and $Y$ is a constant, it follows from the N-P Lemma that $\phi$ is strictly monotone in $\theta$. However, for any $\theta > \theta_0$, consider another test $\phi_2(x) = 1\{\sum_{i=1}^n T(x_i) > c_2\}$ where $c_2$ satisfies $\alpha = \P(\sum_{i=1}^n T(X_i) > c_2 | \theta = \theta_0)$, and it follows from the N-P Lemma that $\phi_2$ is more powerful than $\phi_1$. This is a contradiction.

5. (optional) Let $X_1, \ldots, X_n$ be a random sample from $\text{Unif}(\theta, \theta + 1)$ distribution. To test $H_0 : \theta = 0$ versus $H_1 : \theta > 0$, use the test

$$\text{reject } H_0 \text{ if } Y_n \geq 1 \text{ or } Y_1 \geq k,$$

where $k$ is a constant, $Y_1 = \min_{1 \leq i \leq n} X_i$ and $Y_n = \max_{1 \leq i \leq n} X_i$. Hint: find the marginal distribution of $Y_1$, and the joint distribution of $(Y_1, Y_n)$.

(a) Determine $k$ so that the test has size $\alpha$.

**Solution:** It follows from $\alpha = \P(Y_1 \geq k \text{ or } Y_n \geq 1 | \theta = 0) = (1 - k)^n$ that $k = 1 - \alpha^{1/n}$.

(b) Find an expression for the power function of the test in part (a).

**Solution:** The PDF of $Y_1, Y_n$ are

$$f_{Y_1}(y) = n(\theta + 1 - y)^{n-1}, \quad f_{Y_n}(y) = n(y - \theta)^{n-1}, \quad \theta \leq y \leq \theta + 1,$$

and the joint PDF of $(Y_1, Y_n)$ is

$$f_{Y_1, Y_n}(y_1, y_n) = n(n - 1)(y_n - y_1)^{n-2}, \quad \theta \leq y_1 \leq y_n \leq \theta + 1.$$

If $0 < \theta \leq k$,

$$\P(Y_1 \geq k \text{ or } Y_n \geq 1 | \theta) = \P(Y_1 \geq k | \theta) + \P(Y_n \geq 1 | \theta) - \P(Y_1 \geq k, Y_n \geq 1 | \theta)$$

$$= \P(Y_1 \geq k | \theta) + \P(Y_1 < k, Y_n \geq 1 | \theta)$$

$$= 1 + \alpha - (1 - \theta)^n.$$

If $\theta \geq k$,

$$\P(Y_1 \geq k \text{ or } Y_n \geq 1 | \theta) = 1.$$