Math 281C Homework 2

Due: 5:00pm, April 15th

1. Suppose X is one observation from a population with beta$(\theta, 1)$ pdf—$Cx^{\theta-1}$ for $0 < x < 1$.
   (a) For testing $H_0 : \theta \le 1$ versus $H_1 : \theta > 1$, find the size and sketch the power function of the test that rejects H_0 if $X > 1/2$.
   (b) Find the most powerful level α test of $H_0 : \theta = 1$ versus $H_1 : \theta = 2$.
   (c) Is there a UMP test of $H_0 : \theta \le 1$ versus $H_1 : \theta > 1$? If so, find it; if not, prove so.

2. Let X be one observation from a Cauchy scale distribution with density

$$f_\theta(x) = \frac{\theta}{\pi}\frac{1}{\theta^2 + x^2}, \quad -\infty < x < \infty, \theta > 0.$$

   (a) Show that this family does not have an MLR in x.
   (b) Show that the distribution of $|X|$ does have an MLR.

3. Let X be one observation from a Cauchy distribution

$$f_\theta(x) = \frac{C}{1 + (x - \theta)^2}, \quad x \in \mathbb{R}.$$

   (a) Show that this family does not have an MLR in x.
   (b) Show that the test

$$\phi(x) = \mathbb{1}(1 < x < 3)$$

   is UMP of its size for testing $H_0 : \theta = 0$ versus $H_1 : \theta = 1$. Calculate the Type I and type II error probabilities.

4. Let $X_1, \ldots, X_n$ be i.i.d. from the exponential family

$$f_\theta(\mathbf{x}) = \exp\{\eta(\theta)T(\mathbf{x}) - \xi(\theta)\}h(\mathbf{x}),$$

   where $\eta(\theta)$ is strictly monotone in θ. Show that UMP tests do not exist for testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta \ne \theta_0$. Hint: Examine the example in Section 4.2.

5. (optional) Let $X_1, \ldots, X_n$ be a random sample from Unif$(\theta, \theta+1)$ distribution. To test $H_0 : \theta = 0$ versus $H_1 : \theta > 0$, use the test

$$\text{reject } H_0 \quad \text{if } Y_n \ge 1 \text{ or } Y_1 \ge k,$$

   where k is a constant, $Y_1 = \min_{1 \le i \le n} X_i$ and $Y_n = \max_{1 \le i \le n} X_i$. Hint: find the marginal distribution of Y_1, and the joint distribution of (Y_1, Y_n).
   (a) Determine k so that the test has size α.
   (b) Find an expression for the power function of the test in part (a).