Math 281C Homework 1 Solutions

1. The random variable X has p.d.f $f(x) = e^{-x}$, $x > 0$. One observation is obtained on the random variable $Y = X^\theta$, and a test of $H_0 : \theta = 1$ versus $H_1 : \theta = 2$ needs to be constructed. Find the UMP level $\alpha = 0.1$ test, and compute the Type II error probability.

   Solution: Since $Y = X^\theta$, for any $y > 0$, we have
   $$\mathbb{P}(Y \le y) = \mathbb{P}(X \le y^{1/\theta}) = 1 - e^{-y^{1/\theta}},$$
   and the PDF of Y is
   $$f_Y(y) = \frac{1}{\theta} e^{-y^{1/\theta}} y^{1/\theta - 1}.$$
   By the Neyman–Pearson Lemma, the UMP test rejects H_0 if
   $$\lambda(y) = \frac{f(y|\theta = 2)}{f(y|\theta = 1)} = \frac{1}{2} y^{-1/2} e^{y - y^{1/2}} > C$$
   for some constant C. It can be checked that $\lambda(y)$ is decreasing for $y \in (0,1)$ and increasing for $y \in (1, \infty)$, so it is equivalent to reject H_0 if $y < c_0$ or $y > c_1$, we then solve c_0 and c_1 numerically. Combining
   $$0.1 = \alpha = \mathbb{P}(Y < c_0|\theta = 1) + \mathbb{P}(Y > c_1|\theta = 1) = 1 - e^{-c_0} + e^{-c_1}$$
   and $\lambda(c_0) = \lambda(c_1)$ gives us $c_0 = 0.076546$ and $c_1 = 3.637798$. The above set of equations can be numerically solved by writing an R program.
   Consequently, the Type II error probability is
   $$\mathbb{P}(c_0 \le Y \le c_1|\theta = 2) = e^{-c_0^{1/2}} - e^{-c_1^{1/2}} = 0.609824.$$

2. Let X_1, X_2 be iid Uniform$(\theta, \theta + 1)$. For testing $H_0 : \theta = 0$ versus $H_1 : \theta > 0$, we have two competing tests
   $$\phi_1(X_1) : \text{Reject } H_0 \text{ if } X_1 > 0.95,$$
   $$\phi_2(X_1, X_2) : \text{Reject } H_0 \text{ if } X_1 + X_2 > C.$$

   (a) Find the value C so that ϕ_2 has the same size as ϕ_1.

   Solution: For $\phi_1(X_1)$, $\alpha = \mathbb{P}(X_1 > 0.95|\theta = 0) = 0.05$, then for $\phi_2(X_1, X_2)$, the density of $X_1 + X_2$ can be derived based on convolution,
   $$f_{X_1+X_2}(x|\theta = 0) = \begin{cases} x & \text{if } 0 \le x \le 1 \\ 2 - x & \text{if } 1 < x \le 2 \\ 0 & \text{otherwise,} \end{cases}$$
   and $0.05 = \alpha = \mathbb{P}(X_1 + X_2 > C|\theta = 0) = (2 - C)^2/2$, which gives us $C = 1.68$.

   (b) Calculate the power function of each test. Draw a well-labeled graph of each power function.

   Solution: For $\phi_1(X_1)$,
   $$\beta_1(\theta) = \mathbb{P}(X_1 > 0.95|\theta) = \begin{cases} \theta + 0.05 & \text{if } 0 < \theta \le 0.95 \\ 1 & \text{if } \theta > 0.95. \end{cases}$$
   For $\phi_2(X_1, X_2)$, the density of $X_1 + X_2$ under $\theta > 0$ can be similarly obtained, and
   $$\beta_2(\theta) = \mathbb{P}(X_1 + X_2 > C|\theta) = \begin{cases} (2\theta + 2 - C)^2/2 & \text{if } 0 < \theta \le (C - 1)/2 \\ 1 - (C - 2\theta)^2/2 & \text{if } (C - 1)/2 < \theta \le C/2 \\ 1 & \text{if } \theta > C/2. \end{cases}$$

(c) Prove or disprove: ϕ_2 is a more powerful test that ϕ_1.

Solution: No. ϕ_1 is more powerful when θ is near 0.

(d) Show how to get a test that has the same size but is more powerful than ϕ_2.

Solution: Consider the new test

$$\phi_3(X_1, X_2): \text{ Reject } H_0 \text{ if } X_1 + X_2 > C \text{ or } X_1 > 1 \text{ or } X_2 > 1.$$

Under H_0, $\mathbb{P}(X_1 > 1) = \mathbb{P}(X_2 > 1) = 0$, so ϕ_3 has the same size as ϕ_2, but the reject region of ϕ_3 is larger than the reject region of ϕ_2, so it is more powerful.

3. Show that for a random sample $X_1, \ldots, X_n$ from a $N(0, \sigma^2)$ population, the most powerful test of $H_0 : \sigma = \sigma_0$ versus $H_1 : \sigma = \sigma_1$, where $\sigma_0 < \sigma_1$, is given by

$$\phi(T) = \begin{cases} 1 & \text{if } T > c \\ 0 & \text{if } T \le c, \end{cases}$$

where $T = T(X_1, \ldots, X_n) = \sum_{i=1}^n X_i^2$. For a given value of α (the size of the Type I error), show how the value of c is explicitly determined.

Solution: By the Neyman–Pearson Lemma, the UMP test rejects H_0 if

$$\begin{aligned} \lambda(X_1, \ldots, X_n) &= \frac{f(X_1, \ldots, X_n | \sigma = \sigma_1)}{f(X_1, \ldots, X_n | \sigma = \sigma_0)} \\ &= \frac{\sigma_0^n}{\sigma_1^n} \exp\left\{ T\left(\frac{1}{2\sigma_0^2} - \frac{1}{2\sigma_1^2} \right) \right\} > k \end{aligned}$$

for some constant k. Since the above function is increasing in T, so it is equivalent that we reject H_0 if $T > c$.

To determine the value of c, notice that under $\sigma = \sigma_0$, $\sum_{i=1}^n X_i^2/\sigma_0^2 \sim \chi_n^2$, so

$$\alpha = \mathbb{P}(T > c | \sigma = \sigma_0) = \mathbb{P}(\chi_n^2 > c/\sigma_0^2),$$

which gives $c = \sigma_0^2 \chi_{n,\alpha}^2$.