Math 281C Homework 7

Due: June 8, 5 pm

1. Define

$$r_S(\lambda, \mu) = \mathbb{E}[\text{soft}_\lambda(y) - \mu]^2,$$

where $y \sim \mathcal{N}(\mu, 1)$. Show that

(a) $\mu \to r_S(\lambda, \mu)$ is increasing on $[0, \infty)$;

(b) For all $\lambda > 0$, we have

$$r_S(\lambda, 0) \le \sqrt{\frac{2}{\pi}} \frac{1}{\lambda} e^{-\lambda^2/2};$$

(c) When μ approaches $\pm\infty$,

$$\lim_{\mu \to \infty} r_S(\lambda, \mu) = 1 + \lambda^2,$$

and

$$\sup_{\mu \in \mathbb{R}} r_S(\lambda, \mu) = 1 + \lambda^2.$$

2. Consider the model

$$Y = \theta + \sigma\varepsilon,$$

where $\varepsilon \in \mathbb{R}^n$ consists of independent mean-zero 1-sub-Gaussian components, and assume θ is k-sparse:

$$\|\theta\|_0 = \text{card}\{j \in [n] : \theta_j \ne 0\} \le k.$$

In this question, we investigate the soft-thresholding estimator

$$\widehat{\theta} := \operatorname*{argmin}_\theta \frac{1}{2} \|\theta - Y\|_2^2 + \lambda \|\theta\|_1.$$

(a) Show that if $\lambda \ge \sigma \|\varepsilon\|_\infty$, then

$$\|\widehat{\theta} - \theta\|_2^2 \le 4k\lambda^2;$$

(b) Show that if

$$\lambda = 2\sqrt{\sigma^2 \log(2n)},$$

then with probability at least $1 - 1/(2n)$,

$$\|\widehat{\theta} - \theta\|_2^2 \le 16k\sigma^2 \log(2n).$$