Math 281C Homework 3

Due: May 1, 5 pm

1. Let $\mathcal{H}_k$ denote the indicators of all closed half-spaces in $\mathbb{R}^k$, i.e. $\mathcal{H}_k = \{x \mapsto I(\langle a, x\rangle + b \leq 0) : a \in \mathbb{R}^k, b \in \mathbb{R}\}$. Show that the VC dimension of $\mathcal{H}_k$ is exactly equal to $k+1$.

2. Consider the sphere $S_{a,b} = \{x \in \mathbb{R}^k : \|x - a\|_2 \leq b\}$, where $(a, b) \in \mathbb{R}^k \times \mathbb{R}_+$ specify its center and radius, respectively. Define the function $f_{a,b}(x) = \|x\|_2^2 - 2\sum_{j=1}^k a_j x_j + \|a\|_2^2 - b^2$, so that $S_{a,b} = \{x \in \mathbb{R}^k : f_{a,b}(x) \leq 0\}$. Let $\mathcal{S}_k = \{x \mapsto I\{f_{a,b}(x) \leq 0\} : a \in \mathbb{R}^k, b \geq 0\}$. Show that the VC dimension of $\mathcal{S}_k$ is at most $k+2$.

3. Consider the class of all spheres in $\mathbb{R}^2$: $\mathcal{S}_2$, where $\mathcal{S}_k$ is defined in question 2.
   (a) Show that $\mathcal{S}_2$ can shatter any subset of three points that are not collinear.
   (b) Conclude that the VC dimension of $\mathcal{S}_2$ is 3.