Math 281C Homework 2 Solutions

1. Suppose that $X \geq 0$ and that the moment generating function of X exists in an interval around zero. Given some $\delta > 0$ and integer $k = 0, 1, 2, \ldots$, show that

$$\inf_{k=0,1,2,\ldots} \frac{\mathbb{E}[|X|^k]}{\delta^k} \leq \inf_{\lambda>0} \frac{\mathbb{E}[e^{\lambda X}]}{e^{\lambda\delta}}.$$

Solution: We'll use the following inequality to complete the proof. For two positive sequences of real numbers $\{a_i\}_{i=0}^n$ and $\{b_i\}_{i=0}^n$, if $\sum_{i=0}^\infty a_i$ and $\sum_{i=0}^\infty b_i$ exist, we have

$$\inf_{i=0,1,2,\ldots} \frac{a_i}{b_i} \leq \frac{\sum_{i=0}^\infty a_i}{\sum_{i=0}^\infty b_i}. \tag{1}$$

Now, suppose the moment generating function of X exists in $[0, b]$. If $\lambda > b$, then $\mathbb{E}[e^{\lambda X}] = \infty$; if $\lambda \leq b$, applying Taylor expansion and (1), we have that for any $0 < \lambda \leq b$,

$$\frac{\mathbb{E}[e^{\lambda X}]}{e^{\lambda\delta}} = \frac{\sum_{i=0}^\infty \lambda^i \mathbb{E}[|X|^i]/i!}{\sum_{i=0}^\infty \lambda^i \delta^i/i!} \geq \inf_{i=0,1,2,\ldots} \frac{\mathbb{E}[|X|^i]}{\delta^i}.$$

Taking infimum over λ completes the proof.

Remark: An optimized Markov moments bound is always at least as good as the Chernoff bound.

2. Let $\{X_i\}_{i=1}^n$ be i.i.d. random variables drawn from a density f on the real line. Recall from Math 281A that a standard estimate of f is the kernel density estimate

$$\widehat{f}_n(x) := \frac{1}{nh} \sum_{i=1}^n K\left(\frac{x - X_i}{h}\right),$$

where $K : \mathbb{R} \to [0, \infty)$ is a kernel function satisfying $\int_{-\infty}^\infty K(t)\mathrm{d}t = 1$, and $h > 0$ is a bandwidth parameter. Suppose that we assess the quality of $\widehat{f}_n$ using the ℓ_1-norm

$$||\widehat{f}_n - f||_1 := \int_{-\infty}^\infty |\widehat{f}_n(t) - f(t)|\mathrm{d}t.$$

Prove that

$$\mathbb{P}(||\widehat{f}_n - f||_1 \geq \mathbb{E}[||\widehat{f}_n - f||_1] + \delta) \leq e^{-n\delta^2/2}.$$

Solution: Denote $g(X_1, \cdots, X_i, \cdots, X_n) = ||\widehat{f}_n - f||_1$, we have

$$\begin{aligned} &|g(X_1, \cdots, X_i, \cdots, X_n) - g(X_1, \cdots, X_i', \cdots, X_n)| \\ \leq &\frac{1}{n} \int_{-\infty}^\infty \left| \frac{1}{h} K\left(\frac{t - X_i}{h}\right) - \frac{1}{h} K\left(\frac{t - X_i'}{h}\right) \right| \mathrm{d}t \leq \frac{2}{n}. \end{aligned}$$

Applying the one-sided bounded difference inequality gives

$$\mathbb{P}(||\widehat{f}_n - f||_1 \geq \mathbb{E}[||\widehat{f}_n - f||_1] + \delta) \leq e^{-2\delta^2/(4/n)} = e^{-n\delta^2/2}.$$