Math 281C Homework 1 Solution

1. Let $U \in \mathbb{R}$ be a random variable such that $\mathbb{E}[U] = 0$ and $a \le U \le b$ almost surely, for some constants $b \ge a$. Prove that for any $\lambda \ge 0$,

$$\Psi_U(\lambda) := \log \mathbb{E} e^{\lambda U} \le (b-a)^2 \lambda^2 / 8.$$

Solution: By Taylor expansion around 0,

$$\Psi_U(\lambda) = \Psi_U(0) + \lambda \Psi_U'(0) + \lambda^2 \Psi_U''(\lambda')/2,$$

for some $0 \le \lambda' \le \lambda$. It can be calculated that $\Psi_U(0) = \Psi_U'(0) = 0$, and

$$\Psi_U''(\lambda) = \mathbb{E}_\lambda[U^2] - (\mathbb{E}_\lambda[U])^2,$$

where

$$\mathbb{E}_\lambda[f(U)] = \mathbb{E}[f(U)e^{\lambda U}]/\mathbb{E}[e^{\lambda U}],$$

for any $\lambda \ge 0$. This fact means that $\Psi_U''(\lambda)$ is the variance of another random variable that is also bounded on $[a,b]$ (see details in Lecture 3), so $\Psi_U''(\lambda) \le (b-a)^2/4$. This completes the proof.

Remark: Bounded random variable $X \in [a,b]$ is sub-Gaussian with parameter $(b-a)/2$.

2. Let $Z \sim \mathcal{N}(0,1)$. Prove that for any $t > 0$,

$$\frac{t}{\sqrt{2\pi}(1+t^2)} e^{-t^2/2} \le \mathbb{P}(Z \ge t) \le \frac{1}{\sqrt{2\pi} t} e^{-t^2/2}.$$

Solution: For the upper bound, by a change of variable $x = y + t$, we have

$$\begin{aligned}
\mathbb{P}(Z \ge t) &= \frac{1}{\sqrt{2\pi}} \int_t^\infty e^{-x^2/2} \mathrm{d}x = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-t^2/2} e^{-ty} e^{-y^2/2} \mathrm{d}y \\
&\le \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \int_0^\infty e^{-ty} \mathrm{d}y = \frac{1}{\sqrt{2\pi} t} e^{-t^2/2}.
\end{aligned}$$

For the lower bound, notice that

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{t}{1+t^2} e^{-t^2/2} = \frac{1 - 2t^2 - t^4}{(1+t^2)^2} e^{-t^2/2}.$$

So,

$$\begin{aligned}
\frac{t}{\sqrt{2\pi}(1+t^2)} e^{-t^2/2} &= \frac{1}{\sqrt{2\pi}} \int_t^\infty \frac{x^4 + 2x^2 - 1}{(1+x^2)^2} e^{-x^2/2} \mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} \int_t^\infty 1 - \frac{2}{(1+x^2)^2} e^{-x^2/2} \mathrm{d}x \le \frac{1}{\sqrt{2\pi}} \int_t^\infty e^{-x^2/2} \mathrm{d}x = \mathbb{P}(Z \ge t).
\end{aligned}$$

This completes the proof.

Remark: In particular, when $t \ge 1$, the tail bound of $Z \sim \mathcal{N}(0,1)$ is $\mathbb{P}(Z \ge t) \lesssim e^{-t^2/2}$.

3. Consider the function

$$h(u) = (1+u)\log(1+u) - u,$$

where $u > -1$. Prove the for any $u \ge 0$,

$$h(u) \ge \frac{u^2}{2(1+u/3)}.$$

Solution: It's equivalent to show that

$$f(u) := 2(1+u)(1+u/3)\log(1+u) - 5u^2/3 - 2u \geq 0.$$

It can be shown that

$$f'(u) = (4u/3 + 8/3)\log(1+u) - 8u/3 \geq 0,$$

and $f(0) = 0$. This completes the proof.

4. Assume $\{\xi_i, \mathcal{F}_i\}_{i=1}^n$ is a martingale difference sequence, where $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \mathcal{F}_n$ are σ-fields. That is, each ξ_i is $\mathcal{F}_i$-measurable and $\mathbb{E}[\xi_i | \mathcal{F}_{i-1}] = 0$ almost surely. Moreover, the conditional distribution of ξ_n given $\mathcal{F}_{n-1}$ is supported on an interval with width bounded by R_n. Show that

$$\mathbb{E}[e^{\lambda \xi_n} | \mathcal{F}_{n-1}] \leq e^{\lambda^2 R_n^2/8}.$$

Solution: We define

$$\Psi_{\xi_n}(\lambda) := \log \mathbb{E}[e^{\lambda \xi_n} | \mathcal{F}_{n-1}].$$

By Taylor expansion around 0,

$$\Psi_{\xi_n}(\lambda) = \Psi_{\xi_n}(0) + \lambda \Psi'_{\xi_n}(0) + \lambda^2 \Psi''_{\xi_n}(\lambda')/2,$$

for some $0 \leq \lambda' \leq \lambda$. Using exactly the same argument as in question 1 gives the desired result.