Math 281C Homework 1

Due: Apr 17, 5 pm

1. Let $U \in \mathbb{R}$ be a random variable such that $\mathbb{E}[U] = 0$ and $a \le U \le b$ almost surely, for some constants $b \ge a$. Prove that for any $\lambda \ge 0$,
$$\Psi_U(\lambda) := \log \mathbb{E} e^{\lambda U} \le (b-a)^2 \lambda^2/8.$$

2. Let $Z \sim \mathcal{N}(0,1)$. Prove that for any $t > 0$,
$$\frac{t}{\sqrt{2\pi}(1+t^2)} e^{-t^2/2} \le \mathbb{P}(Z \ge t) \le \frac{1}{\sqrt{2\pi}t} e^{-t^2/2}.$$

3. Consider the function
$$h(u) = (1+u)\log(1+u) - u,$$
where $u > -1$. Prove the for any $u \ge 0$,
$$h(u) \ge \frac{u^2}{2(1+u/3)}.$$

4. Assume $\{\xi_i, \mathcal{F}_i\}_{i=1}^n$ is a martingale difference sequence, where $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \mathcal{F}_n$ are σ-fields. That is, each ξ_i is $\mathcal{F}_i$-measurable and $\mathbb{E}[\xi_i | \mathcal{F}_{i-1}] = 0$ almost surely. Moreover, the conditional distribution of ξ_n given $\mathcal{F}_{n-1}$ is supported on an interval with width bounded by R_n. Show that
$$\mathbb{E}[e^{\lambda \xi_n} | \mathcal{F}_{n-1}] \le e^{\lambda^2 R_n^2/8}.$$