Math 281A Homework 5 Solution

1. Let $\{x_i\}_{i=1}^n$ be i.i.d. sample from a strictly positive density that is symmetric about θ, show that the Huber M-estimator for location is consistent for θ.

   Solution: Huber M-estimator $\hat{\mu}_n$ is the solution of

$$\Psi_n(\mu) := \frac{1}{n}\sum_{i=1}^n \psi(X_i - \mu) = 0,$$

   where

$$\psi(x) = \begin{cases} -\tau & \text{if } x < -\tau \\ x & \text{if } -\tau \le x \le \tau \\ \tau & \text{if } x > \tau \end{cases}.$$

   Define

$$\Psi(\mu) := \mathbb{E}[\psi(X - \mu)],$$

   then by weak law of large numbers, $\Psi_n(\mu) \xrightarrow{p} \Psi(\mu)$ for every μ. Moreover, $\Psi_n(\mu)$ is nonincreasing in μ, and it remains to check that for any $\epsilon > 0$,

$$\Psi(\theta - \epsilon) > 0 > \Psi(\theta + \epsilon).$$

   For the first inequality,

$$\begin{aligned}\Psi(\theta - \epsilon) &= \int_{-\infty}^{\infty} \psi(x - (\theta - \epsilon)) f(x)\mathrm{d}x \\ &= \int_{-\infty}^{\infty} \psi(x) f(x + (\theta - \epsilon))\mathrm{d}x = \int_{-\infty}^{\infty} \psi(x) g(x)\mathrm{d}x,\end{aligned}$$

   where $g(x)$ is symmetric about ϵ. Then

$$\begin{aligned}\int_{-\infty}^{\infty} \psi(x)g(x)\mathrm{d}x &> \int_{-\infty}^{0} \psi(x)g(x)\mathrm{d}x + \int_{2\epsilon}^{\infty} \psi(x)g(x)\mathrm{d}x \\ &= \int_{-\infty}^{0} [\psi(2\epsilon - x) + \psi(x)]g(x)\mathrm{d}x \ge 0.\end{aligned}$$

   The second inequality follows similarly. Together, they guarantee that $\hat{\mu}_n \xrightarrow{p} \theta$.

2. Let $\{x_i\}_{i=1}^n$ be i.i.d. sample from a strictly positive density. Define

$$\psi(x) = \frac{2}{1 + e^{-x}} - 1,$$

   and $\hat{\theta}_n$ be the solution of

$$\sum_{i=1}^n \psi(X_i - \theta) = 0.$$

   (a) Show that $\hat{\theta}_n \xrightarrow{P} \theta_0$ for some θ_0, and express θ_0 in the density of observations;

   Solution: Notice that $\psi(x) = \tanh(x/2)$, and if we define

$$\Psi(\theta) := \mathbb{E}[\psi(X - \theta)] = \int_{-\infty}^{\infty} \tanh((x - \theta)/2) f(x)\mathrm{d}x,$$

then $\Psi(\theta)$ is strictly decreasing with $\Psi(-\infty) = 1$ and $\Psi(\infty) = -1$, so there is a unique solution of $\Psi(\theta) = 0$, and we denote the solution as θ_0. If we further define

$$\Psi_n(\theta) = \frac{1}{n} \sum_{i=1}^{n} \psi(X_i - \theta),$$

then we have $\Psi_n(\theta) \xrightarrow{p} \Psi(\theta)$ and $\Psi(\theta - \epsilon) > 0 > \Psi(\theta + \epsilon)$ for every $\epsilon > 0$, which implies $\hat{\theta}_n \xrightarrow{P} \theta_0$.

(b) Show that $\sqrt{n}(\hat{\theta}_n - \theta_0)$ converges in distribution and find the limit variance.

Solution: It can be shown that $\psi(x)$ is twice continuously differentiable with

$$\psi'(x) = \tanh'(x/2) = \frac{\operatorname{sech}^2(x/2)}{2}$$

and

$$\psi''(x) = \tanh''(x/2) = -\frac{\operatorname{sech}^2(t/2)\tanh(t/2)}{2}.$$

Furthermore, we have $|\psi'(x)| \le 1/2$, $|\psi''(x)| \le 1/2$, and $\psi'(x) > 0$. Together, they give us

$$\sqrt{n}(\hat{\theta}_n - \theta_0) \xrightarrow{d} N(0, \sigma^2),$$

where

$$\sigma^2 = \frac{4\mathbb{E}[\tanh^2((X - \theta)/2)]}{(\mathbb{E}[\operatorname{sech}^2((X - \theta)/2)])^2}.$$

3. Let $\{x_i\}_{i=1}^n$ be i.i.d. sample from Uniform$(0,1)$, determine the relative efficiency of the sample median and the sample mean.

Solution: For $X \sim$ Uniform$(0,1)$, we have $\mathbb{E}[X] = 1/2$ and $\operatorname{var}(X) = 1/12$, so the asymptotic variance of sample mean is $1/12$. Besides, the population median is also $1/2$ and the asymptotic variance of sample median is $1/(4f^2(\theta_0)) = 1/4$. Hence, the relative efficiency is $1/3$.

4. Let $\{x_i\}_{i=1}^n$ be i.i.d. sample from $N(\theta, 1)$, find the relative efficiency of the Huber estimator and the sample mean.

Solution: The asymptotic variance of sample mean is 1, and the asymptotic variance of Huber estimator is $\mathbb{E}[\psi_\theta^2]/(\mathbb{E}[\psi_\theta'])^2$. Then we compute these two items,

$$\mathbb{E}[\psi_\theta^2] = 2\int_0^\tau x^2\phi(x)\mathrm{d}x + 2\tau^2 \int_\tau^\infty \phi(x)\mathrm{d}x,$$

and

$$\mathbb{E}[\psi_\theta'] = \int_{-\tau}^{\tau} \phi(x)\mathrm{d}x,$$

where $\phi(x)$ denotes the density of standard normal distribution. So the relative efficiency is

$$\frac{(\int_{-\tau}^{\tau} \phi(x)\mathrm{d}x)^2}{2\int_0^\tau x^2\phi(x)\mathrm{d}x + 2\tau^2\int_\tau^\infty \phi(x)\mathrm{d}x}.$$