# Math 281A Homework 3 Solution

1. The Pareto distribution possesses the density function

$$f_{\alpha,\mu}(x) = \frac{\alpha\mu^\alpha}{x^{\alpha+1}} I\{x \geq \mu\}$$

with $\alpha, \mu > 0$. Denote $\hat{\alpha}_n$ as the MLE of $\alpha$ based on an i.i.d. sample $\{X_i\}_{i=1}^n$. Determine the limit distribution of $\sqrt{n}(\hat{\alpha}_n - \alpha)$ when

(a) $\mu$ is known;

**Solution:** Computational details are omitted. When $\mu$ is known, the MLE of $\alpha$ is

$$\hat{\alpha}_n = 1\Big/\overline{\log(X/\mu)},$$

where

$$\overline{\log(X/\mu)} = \frac{\sum_{i=1}^n \log(X_i/\mu)}{n}.$$

It can be calculated from density function that for $X \sim f_{\alpha,\mu}(x)$,

$$\mathbb{E}[\log(X/\mu)] = \frac{1}{\alpha}, \text{ and } \mathrm{Var}(\log(X/\mu)) = \frac{1}{\alpha^2}.$$

By central limit theorem, we have

$$\sqrt{n}\big(\overline{\log(X/\mu)} - 1/\alpha\big) \xrightarrow{d} N(0, 1/\alpha^2).$$

Applying delta method gives us

$$\sqrt{n}(\hat{\alpha}_n - \alpha) \xrightarrow{d} N(0, \alpha^2).$$

(b) $\mu$ is unknown.

**Solution:** When $\mu$ is unknown, the MLEs of $\alpha$ and $\mu$ are

$$\hat{\alpha}_n = \frac{1}{\overline{\log X} - \log X_{(1)}}, \text{ and } \hat{\mu}_n = X_{(1)}.$$

In part (a), we know that

$$\sqrt{n}\big(\overline{\log X} - \log\mu - 1/\alpha\big) \xrightarrow{d} N(0, 1/\alpha^2). \tag{1}$$

Then, for any $u > 0$,

$$\mathbb{P}(\sqrt{n}(\log X_{(1)} - \log\mu) \geq u) = \mathbb{P}(X_{(1)} \geq \mu e^{u/\sqrt{n}})$$
$$= (\mathbb{P}(X \geq \mu e^{u/\sqrt{n}}))^n = e^{-\sqrt{n}\alpha u} \to 0.$$

Since $\log X_{(1)} - \log\mu \geq 0$, this further implies

$$\sqrt{n}(\log X_{(1)} - \log\mu) \xrightarrow{P} 0. \tag{2}$$

Subtracting (2) from (1) with Slutsky theorem yields

$$\sqrt{n}\big(\overline{\log X} - \log X_{(1)} - 1/\alpha\big) \xrightarrow{d} N(0, 1/\alpha^2).$$

Finally, applying delta method similarly as in part (a), we obtain

$$\sqrt{n}(\hat{\alpha}_n - \alpha) \xrightarrow{d} N(0, \alpha^2).$$

**Remark:** If $X \sim \text{Pareto}(\alpha, \mu)$, then $\log(X/\mu) \sim \exp(\alpha)$.

2. Let $\{X_i\}_{i=1}^n$ be an i.i.d. sample from Poisson distribution with mean $\theta$. Find a variance stabilizing transformation for the sample mean and construct a confidence interval for $\theta$ based on this.

   **Solution:** This question is postponed to next week.

3. Suppose $\phi : \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable at $\theta$ with $\phi'(\theta) = 0$ and $\phi''(\theta) \neq 0$. Suppose that $\sqrt{n}(T_n - \theta) \xrightarrow{d} N(0,1)$.

   (a) Show that $\sqrt{n}(\phi(T_n) - \phi(\theta)) \xrightarrow{P} 0$;

   **Solution:** First off, notice that

   $$T_n - \theta = \frac{1}{\sqrt{n}}\sqrt{n}(T_n - \theta) \xrightarrow{P} 0.$$

   Then we define

   $$g(h) = \begin{cases} \frac{\phi(\theta+h)-\phi(\theta)}{|h|} & \text{if } h \neq 0, \\ 0 & \text{if } h = 0. \end{cases}$$

   By the differentiability of $\phi(\cdot)$ at $\theta$, $g(h)$ is continuous at 0. Hence, by continuous mapping theorem, $g(T_n - \theta) \xrightarrow{P} 0$. Combining this result with Slutsky theorem gives

   $$\sqrt{n}|T_n - \theta| g(T_n - \theta) \xrightarrow{P} 0,$$

   which is equivalent as

   $$\sqrt{n}(\phi(T_n) - \phi(\theta)) \xrightarrow{P} 0,$$

   using the definition of $g(h)$ with $h = T_n - \theta$.

   (b) What is the asymptotic distribution of $n(\phi(T_n) - \phi(\theta))$? Justify your answer.

   **Solution:** Now we define

   $$g(h) = \begin{cases} \frac{\phi(\theta+h)-\phi(\theta)-\phi''(\theta)(T_n-\theta)^2/2}{h^2} & \text{if } h \neq 0, \\ 0 & \text{if } h = 0. \end{cases}$$

   With similar technique used in part (a), it can be established that $g(h)$ is continuous at 0, and

   $$n(T_n - \theta)^2 g(T_n - \theta) \xrightarrow{P} 0,$$

   which further implies

   $$n(\phi(T_n) - \phi(\theta) - \phi''(\theta)(T_n - \theta)^2/2) \xrightarrow{P} 0. \tag{3}$$

   On the other hand, by continuous mapping theorem, we have

   $$n\phi''(\theta)\frac{(T_n - \theta)^2}{2} \xrightarrow{d} \frac{\phi''(\theta)}{2}\chi_1^2. \tag{4}$$

   Summing up (3) and (4) with Slutsky theorem gives the desired asymptotic distribution:

   $$n(\phi(T_n) - \phi(\theta)) \xrightarrow{d} \frac{\phi''(\theta)}{2}\chi_1^2.$$

4. Let $X_1, \ldots, X_n$ be i.i.d. from density $f_{\lambda,a}(x) = \lambda e^{-\lambda(x-a)} I(x \geq a)$, where $\lambda > 0$ and $a \in \mathbb{R}$ are unknown parameters. Recall that we found the MLE $(\hat{\lambda}_n, \hat{a}_n)$ of $(\lambda, a)$ in homework 1, derive the asymptotic property of $(\hat{\lambda}_n, \hat{a}_n)$.

   **Solution:** From homework 1, we know that

   $$\hat{\lambda}_n = \frac{1}{\bar{X} - X_{(1)}}, \text{ and } \hat{a}_n = X_{(1)}.$$

First, let us derive the marginal limit distributions. For $X \sim f_{\lambda,a}(x)$, we have

$$\mathbb{E}[X] = a + \frac{1}{\lambda}, \text{ and } \mathrm{Var}(X) = \frac{1}{\lambda^2},$$

so by central limit theorem,

$$\sqrt{n}(\bar{X} - (a + 1/\lambda)) \xrightarrow{d} N(0, 1/\lambda^2).$$

With similar technique that leads to (2) in Problem 1, part (b), we have

$$\sqrt{n}(X_{(1)} - a) \xrightarrow{P} 0.$$

Combining the above two display with Slutsky gives

$$\sqrt{n}(\bar{X} - X_{(1)} - 1/\lambda) \xrightarrow{d} N(0, 1/\lambda^2),$$

which further implies the marginal distribution of $\hat{\lambda}_n$ with delta method:

$$\sqrt{n}(\hat{\lambda}_n - \lambda) \xrightarrow{d} N(0, \lambda^2). \tag{5}$$

For the limit distribution of $\hat{a}_n$, consider any $u \geq 0$,

$$\mathbb{P}(n(X_{(1)} - a) \geq u) = \mathbb{P}(X_{(1)} \geq u/n + a) = \left(e^{-\lambda u/n}\right)^n = e^{-\lambda u} \sim \exp(\lambda),$$

therefore,

$$n(\hat{a}_n - a) \xrightarrow{d} \exp(\lambda). \tag{6}$$

To get the joint distribution, it remains to show the independence between $\hat{\lambda}_n$ and $\hat{a}_n$. It can be found that $(X_i - a) \sim \exp(\lambda)$, and if we denote $Y_1 = X_{(1)} - a, Y_i = X_{(i)} - X_{(i-1)}$, for $i = 2, \ldots, n$, then by memoryless property of exponential distribution, $Y_i$'s are mutually independent. Since $\hat{\lambda}_n$ can be written as a function of $\{Y_i\}_{i=2}^n$ and $\hat{a}_n$ is a function of $Y_1$, the independence between $\hat{\lambda}_n$ and $\hat{a}_n$ are justified.

Finally, combining this property with (5) and (6) gives

$$\left(\sqrt{n}(\hat{\lambda}_n - \lambda), n(\hat{a}_n - a)\right) \xrightarrow{d} N(0, \lambda^2) \times \exp(\lambda),$$

where $\times$ denotes measure product.