Math 281A Homework 2

Due: Oct 17, in class

1. Let $X_1, \ldots, X_n$ be i.i.d. from $N(0,1)$, show that $\bar{X}$ and $(X_1 - \bar{X}, \ldots, X_n - \bar{X})$ are independent.

2. Suppose that random vector (X, Y) has probability density function

$$\frac{1}{\pi} e^{-\frac{x^2+y^2}{2}} I(xy > 0).$$

   Does (X, Y) possess a multivariate normal distribution? Find the marginal distributions.

3. Suppose T_n and S_n are sequences of estimators such that

$$\sqrt{n}(T_n - \theta) \xrightarrow{d} N_k(0, \Sigma), \text{ and } S_n \xrightarrow{P} \Sigma,$$

   for a certain vector θ and a nonsingular matrix Σ. Show that

   (a) S_n is nonsingular with probability tending to one;

   (b) $\{\theta : n(T_n - \theta)^\top S_n^{-1}(T_n - \theta) \le \chi^2_{k,\alpha}\}$ is a confidence ellipsoid of asymptotic confidence level $1 - \alpha$.

4. Suppose that $X_m \sim \text{Binomial}(m, p_1), Y_n \sim \text{Binomial}(n, p_2)$ and they are independent. To test $H_0 : p_1 = p_2 = a$, we consider the test statistic

$$C^2_{m,n} = \frac{(X_m - ma)^2}{ma(1-a)} + \frac{(Y_n - na)^2}{na(1-a)}.$$

   (a) Find the limit distribution of $C^2_{m,n}$ as $m, n \to \infty$;

   (b) How would you modify the test statistic if a were unknown? What's the limit distribution after modification? You don't need to rigorously prove this question.