Math 281A Homework 1 Solution

1. Suppose $X_n \sim \text{Binomial}(n, p_n)$, and $\lim_{n\to\infty} np_n = \lambda > 0$. Show that $X_n \xrightarrow{d} \text{Poisson}(\lambda)$.

   Solution: Denote $\lambda_n = np_n$, then we have
$$\begin{aligned}\mathbb{P}(X_n = x) &= \binom{n}{x}\left(\frac{\lambda_n}{n}\right)^x\left(1-\frac{\lambda_n}{n}\right)^{n-x}\\ &= \frac{\lambda_n^x}{x!}\left(1-\frac{\lambda_n}{n}\right)^n \frac{n!}{(n-x)!n^x}\left(1-\frac{\lambda_n}{n}\right)^{-x},\end{aligned}$$
   where
$$\begin{aligned}&\left(1-\frac{\lambda_n}{n}\right)^n \to e^{-\lambda},\\ &\frac{n!}{(n-x)!n^x} = \frac{(n-x+1)\cdots(n-1)n}{n^x} \to 1,\\ &\left(1-\frac{\lambda_n}{n}\right)^{-x} \to 1.\end{aligned}$$
   Combining these results gives
$$\mathbb{P}(X_n = x) \to e^{-\lambda}\frac{\lambda^x}{x!},$$
   and this shows $X_n \xrightarrow{d} \text{Poisson}(\lambda)$.

2. Suppose $X_n \sim \chi_n^2$ with n degrees of freedom. Find a_n and b_n such that
$$\frac{X_n - a_n}{b_n} \xrightarrow{d} N(0,1).$$

   Solution: $a_n = n$, and $b_n = \sqrt{2n}$. The result follows from central limit theorem, since $X_n = \sum_{i=1}^n Z_i^2$, where $\{Z_i\}_{i=1}^n$ are i.i.d. standard normal variables.

3. Let $Y_1, \dots, Y_n$ be i.i.d. samples from Uniform$[0,1]$, and $Y_{(1)}, \dots, Y_{(n)}$ be the order statistics. Show that $n(Y_{(1)}, 1 - Y_{(n)}) \xrightarrow{d} (U, V)$, where U and V are two independent exponential random variables.

   Solution: For any $(u,v) \in [0,n]\times[0,n]$, we have
$$\begin{aligned}\mathbb{P}(nY_{(1)} \ge u, n - nY_{(n)} \ge v) &= \mathbb{P}(Y_{(1)} \ge u/n, Y_{(n)} \le 1 - v/n)\\ &= \prod_{i=1}^n \mathbb{P}(u/n \le Y_i \le 1 - v/n)\\ &= (1-(u+v)/n)^n\\ &\to e^{-(u+v)} = \mathbb{P}(U \ge u, V \ge v),\end{aligned}$$
   where U and V are two independent exponential random variable with parameter 1.

4. Let $X_1, \dots, X_n$ be i.i.d. samples from Uniform$[-\theta, \theta]$, and $X_{(1)}, \dots, X_{(n)}$ be order statistics. Show that the following three statistics are asymptotically consistent estimators of θ.

(a) $X_{(n)}$;

Solution: For any $\epsilon > 0$, we have

$$\mathbb{P}(X_{(n)} \le \theta - \epsilon) = \prod_{i=1}^{n} \mathbb{P}(X_i \le \theta - \epsilon) = \left(1 - \frac{\epsilon}{2\theta}\right)^n \to 0.$$

(b) $-X_{(1)}$;

Solution: For any $\epsilon > 0$, we have

$$\mathbb{P}(-X_{(1)} \le \theta - \epsilon) = \prod_{i=1}^{n} \mathbb{P}(X_i \ge -\theta + \epsilon) = \left(1 - \frac{\epsilon}{2\theta}\right)^n \to 0.$$

(c) $(X_{(n)} - X_{(1)})/2$.

Solution: From (a) and (b), we have

$$X_{(n)}/2 \xrightarrow{P} \theta/2, \text{ and } -X_{(1)}/2 \xrightarrow{P} \theta/2.$$

Hence,

$$(X_{(n)} - X_{(1)})/2 \xrightarrow{P} \theta.$$

5. A random variable X_n is said to follow a t-distribution with n degrees of freedom, if $X_n \sim \sqrt{n}Z/\sqrt{Z_1^2 + \ldots, Z_n^2}$, where $Z, Z_1, \ldots, Z_n$ are i.i.d. from $N(0,1)$. Show that $X_n \xrightarrow{d} N(0,1)$.

Solution: Denote $\tilde{Z} = Z_1^2 + \ldots, Z_n^2$ and we know that $\mathbb{E}[Z_i^2] = 1$, so $\tilde{Z}/n \xrightarrow{P} 1$ by weak law of large numbers. Hence,

$$\sqrt{\frac{\tilde{Z}}{n}} = \sqrt{\frac{Z_1^2 + \ldots, Z_n^2}{n}} \xrightarrow{P} 1.$$

Then by Slutsky theorem, we have

$$Z \Big/ \sqrt{\frac{\tilde{Z}}{n}} \xrightarrow{d} N(0,1).$$

6. Let $X_1, \ldots, X_n$ be i.i.d. from density $f_{\lambda,a}(x) = \lambda e^{-\lambda(x-a)} I(x \ge a)$, where $\lambda \ge 0$ and $a \in \mathbb{R}$ are unknown parameters. Find the MLE $(\hat{\lambda}_n, \hat{a}_n)$ of (λ, a), and show that $(\hat{\lambda}_n, \hat{a}_n) \xrightarrow{P} (\lambda, a)$.

Solution: We omit some details when we calculate MLE,

$$\ell(\lambda, a) = \left(n \log \lambda - \lambda \sum_{i=1}^{n} (X_i - a)\right) I(a \le X_{(1)}).$$

Taking partial derivatives with respect to λ and a gives us

$$\hat{\lambda} = \frac{1}{\bar{X} - X_{(1)}}, \text{ and } \hat{a} = X_{(1)}.$$

Then we show $\hat{\lambda} \xrightarrow{P} \lambda$ and $\hat{a} \xrightarrow{P} a$ separately. From the density function $f_{\lambda,a}(x)$, we can check that for any $\epsilon > 0$,

$$\begin{aligned}
\mathbb{P}(\hat{a} \ge a + \epsilon) &= \prod_{i=1}^{n} \mathbb{P}(X_i \ge a + \epsilon) \\
&= \left(\int_{a+\epsilon}^{\infty} \lambda e^{-\lambda(x-a)} \mathrm{d}x\right)^n \\
&= e^{-n\lambda\epsilon} \to 0.
\end{aligned}$$

So $\hat{a} \xrightarrow{P} a$. Then from the density function again, we can calculate the expectation

$$\mathbb{E}[X] = \int_{a}^{\infty} \lambda x e^{-\lambda(x-a)} \mathrm{d}x = a + \frac{1}{\lambda}.$$

Applying weak law of large numbers yields

$$\bar{X} \xrightarrow{P} a + \frac{1}{\lambda}.$$

Then we have

$$\bar{X} - X_{(1)} \xrightarrow{P} \frac{1}{\lambda},$$

so

$$\hat{\lambda} \xrightarrow{P} \lambda.$$

Combining the previous results gives us

$$(\hat{\lambda}, \hat{a}) \xrightarrow{P} (\lambda, a).$$