Math 281A Homework 1

Due: Oct 10, in class

1. Suppose $X_n \sim \text{Binomial}(n, p_n)$, and $\lim_{n\to\infty} np_n = \lambda > 0$. Show that $X_n \xrightarrow{d} \text{Poisson}(\lambda)$.

2. Suppose $X_n \sim \chi_n^2$ with n degrees of freedom. Find a_n and b_n such that

$$\frac{X_n - a_n}{b_n} \xrightarrow{d} N(0,1).$$

3. Let $Y_1, \ldots, Y_n$ be i.i.d. samples from Uniform$[0,1]$, and $Y_{(1)}, \ldots, Y_{(n)}$ be the order statistics. Show that $n(Y_{(1)}, 1 - Y_{(n)}) \xrightarrow{d} (U, V)$, where U and V are two independent exponential random variables.

4. Let $X_1, \ldots, X_n$ be i.i.d. samples from Uniform$[-\theta, \theta]$, and $X_{(1)}, \ldots, X_{(n)}$ be order statistics. Show that the following three statistics are asymptotically consistent estimators of θ.
   (a) $X_{(n)}$;
   (b) $-X_{(1)}$;
   (c) $(X_{(n)} - X_{(1)})/2$.

5. A random variable X_n is said to follow a t-distribution with n degrees of freedom, if $X_n \sim \sqrt{n}Z/\sqrt{Z_1^2 + \ldots, Z_n^2}$, where $Z, Z_1, \ldots, Z_n$ are i.i.d. from $N(0,1)$. Show that $X_n \xrightarrow{d} N(0,1)$.

6. Let $X_1, \ldots, X_n$ be i.i.d. from density $f_{\lambda,a}(x) = \lambda e^{-\lambda(x-a)}$, when $x \geq a$, where $\lambda > 0$ and $a \in \mathbb{R}$ are unknown parameters. Find the MLE $(\hat{\lambda}_n, \hat{a}_n)$ of (λ, a), and show that $(\hat{\lambda}_n, \hat{a}_n) \xrightarrow{P} (\lambda, a)$.