PROJECTION THEOREMS AND ISOMETRIES OF HYPERBOLIC SPACES

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Abstract. We prove a restricted projection theorem for an $n-2$ dimensional family of projections from $\mathbb{R}^n$ to $\mathbb{R}$.

The family we consider arises naturally in the context of the adjoint representation of the maximal unipotent subgroup of $\mathrm{SO}(n-1,1)$ on the Lie algebra of $\mathrm{SO}(n,1)$.

Contents

1. Introduction

The classical Marstrand projection theorem states that for a compact subset $K \subset \mathbb{R}^n$ and for a.e. $\mathbf{v} \in \mathbb{S}^{n-1}$

$$\dim \mathrm{p}_{\mathbf{v}}(K) = \min(1, \dim K), \tag{1.1}$$

where $\mathrm{p}_{\mathbf{v}}(X) = X \cdot \mathbf{v}$ is the orthogonal projection in the direction of $\mathbf{v}$ and here and in what follows dim denotes the Hausdorff dimension. Analogous statements hold more generally for orthogonal projection into a.e. m-dimensional subspace, with respect to the Lebesgue measure on $\mathrm{Gr}(m,n)$.

The question of obtaining similar results as in (1.1) where $\mathbf{v}$ is confined to a proper Borel subset $B \subset \mathbb{S}^{n-1}$ has also been much studied, e.g., by Mattila, Falconer, Bourgain and others. Note, however, that without further restrictions on B, (1.1) fails: e.g., if

$$B = \{(\cos t, \sin t, 0) : 0 \le t \le 2\pi\}$$

is the great circle in $\mathbb{S}^2$ and K is the z-axis, then $\mathrm{p}_{\mathbf{v}}(K) = 0$ for every $\mathbf{v} \in B$.

It was conjectured by Fässler and Orponen [FO14] that these are essentially the only type of obstructions; more precisely, they conjecture that if

$\gamma : [0,1] \to \mathbb{S}^2$ is a curve so that for all $\{\gamma(t), \gamma'(t), \gamma''(t)\}$ span $\mathbb{R}^3$ for all t, then for a.e. $t \in [0,1]$,

$$\dim \mathrm{p}_{\gamma(t)}(K) = \min(1, \dim K).$$

This conjecture was recently proved by [PYZ22]; see also the earlier work [KOV17] which relies on similar techniques as [PYZ22], and the more recent work [GGW22] which uses different techniques — a major difficulty here is the failure of transversality in sense of [PS00].

In this paper we consider a restricted projection problem in the same vein, which is motivated by recent applications in homogeneous dynamics, see §4 for more details.

Let us fix some notation in order to state the main results. Let $n \geq 3$. We use the following coordinates for $\mathbb{R}^n$

$$\mathbb{R}^n = \{(r_1, \mathbf{w}, r_2) : r_i \in \mathbb{R}, \mathbf{w} \in \mathbb{R}^{n-2}\}.$$

Let $L : \mathbb{R}^{n-2} \to \mathbb{R}^{n-2}$ be an isomorphism, and let $q : \mathbb{R}^{n-2} \to \mathbb{R}$ be a positive definite quadratic form. For every $\mathbf{t} \in \mathbb{R}^{n-2}$, define $\pi_{\mathbf{t}} = \pi_{L,q,\mathbf{t}} : \mathbb{R}^n \to \mathbb{R}$ by

$$\pi_{\mathbf{t}}(r_1, \mathbf{w}, r_2) = r_1 + \mathbf{w} \cdot L(\mathbf{t}) + r_2 q(\mathbf{t})$$

where $\mathbf{w} \cdot L(\mathbf{t})$ is the usual inner product on $\mathbb{R}^{n-2}$.

In this paper, we prove the following theorem.

1.1. **Theorem.** *Let $K \subset \mathbb{R}^n$ be a compact subset. Then for almost every $\mathbf{t} \in \mathbb{R}^{n-2}$ (with resepect to the Lebesgue measure), we have*

$$\dim(\pi_{\mathbf{t}}(K)) = \min(1, \dim K)$$

Indeed for most applications a discretized version of Theorem 1.1 is required. This is the content of the following theorem.

1.2. **Theorem.** *Let $0 < \alpha \leq 1$, and let $0 < \delta_0 \leq 1$. Let $F \subset B_{\mathbb{R}^n}(0,1)$ be a finite set satisfying the following:*

$$\#(B_{\mathbb{R}^n}(X, \delta) \cap F) \leq C \cdot \delta^\alpha \cdot (\#F) \quad \textit{for all } X \in F \textit{ and all } \delta \geq \delta_0$$

where $C \geq 1$.

Let $0 < \varepsilon < \alpha/100$. For every $\delta \geq \delta_0$, there exists a subset $B_\delta \subset B := \{\mathbf{t} \in \mathbb{R}^{n-2} : 1 \leq \|\mathbf{t}\| \leq 2\}$ with

$$|B \setminus B_\delta| \ll \varepsilon^{-A} \delta^\varepsilon$$

so that the following holds. Let $\mathbf{t} \in B_\delta$, then there exists $F_{\delta,\mathbf{t}} \subset F$ with

$$\#(F \setminus F_{\delta,\mathbf{t}}) \ll \varepsilon^{-A} \delta^\varepsilon \cdot (\#F)$$

such that for all $X \in F_{\delta,\mathbf{t}}$, we have

$$\#\big(\{X' \in F_{\delta,\mathbf{t}} : |\pi_{\mathbf{t}}(X') - \pi_{\mathbf{t}}(X)| \leq \delta\}\big) \ll C\delta_0^{-10\varepsilon} \cdot \delta^\alpha \cdot (\#F),$$

where A is absolute and the implied constants depend on L and q.

Remark. Throughout the paper, the notation $a \ll b$ means $a \le Db$ for a positive constant D whose dependence varies and is explicated in different statements.

Also, for a Borel subset $B \subset \mathbb{R}^d$, we denote the Lebesgue measure of B by $|B|$.

The most difficult case of the above theorem is arguably the case $n = 3$, which was studied in [KOV17, PYZ22] using fundamental works of Wolff and Schlag [Wol00, Sch03] — see also [GGW22] for a different approach to a more general problem in the same vein. Indeed when $n > 3$, it is plausible that one may deduce the Theorem 1.2 using techniques of [PS00] more directly; we, however, take a slightly different route which is a hybrid of these two methods.

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2. Proof of Theorem 1.1

Theorem 1.1 can be proved using the finitary version Theorem 1.2. However, for the convenience of the reader, we present a self contained proof of Theorem 1.1 in this brief section. This will also help explain the main idea of the proof of Theorem 1.2.

Let L and q be as in §1. For every $\mathbf{t} \in \mathbb{R}^{n-2}$, define

$$f_{\mathbf{t}} = f_{L,q,\mathbf{t}} : \mathbb{R}^n \to \mathbb{R}^3 \quad \text{by} \quad f_{\mathbf{t}}(r_1, \mathbf{w}, r_2) = \big(r_1, \mathbf{w} \cdot L(\mathbf{t}), r_2 q(\mathbf{t})\big); \tag{2.1}$$

recall our coordinates $\mathbb{R}^n = \{(r_1, \mathbf{w}, r_2) : r_i \in \mathbb{R}, \mathbf{w} \in \mathbb{R}^{n-2}\}$.

2.1. **Lemma.** *There exists $\varepsilon_0 = \varepsilon_0(q, L) \le 0.01$ so that the following holds. Let $0 < \varepsilon < \varepsilon_0$. Let $X, X' \in \mathbb{R}^n$ satisfy that $\|X - X'\| = 1$. Then*

$$\big|\{\mathbf{t} \in B : \big\|f_{\mathbf{t}}(X) - f_{\mathbf{t}}(X')\big\| \le \varepsilon\}\big| \ll \varepsilon$$

where $B = \{\mathbf{t} \in \mathbb{R}^{n-2} : 1 \le \|\mathbf{t}\| \le 2\}$ and the implied constant depend on L and q.

Proof. Let us write $X = (r_1, \mathbf{w}, r_2)$ and $X' = (r_1', \mathbf{w}', r_2')$. Let us denote the set in the lemma by S. If $S \ne \emptyset$, then there exists some $\mathbf{t} \in B$ so that

$$\big\|\big(r_1 - r_1', (\mathbf{w} - \mathbf{w}') \cdot L(\mathbf{t}), (r_2 - r_2')q(\mathbf{t})\big)\big\| \le \varepsilon.$$

Thus $|r_1 - r_1'|, |r_2 - r_2'|\, q(\mathbf{t}) \le \varepsilon \le 1/10$. This and $\|X - X'\| = 1$ imply

$$\big\|\mathbf{w} - \mathbf{w}'\big\| \gg 1$$

where the implied constant depends on $\min_{\|\mathbf{t}\|=1} |q(\mathbf{t})|$.

Altogether, either $S = \emptyset$ in which case the proof is complete, or we may assume $\|\mathbf{w} - \mathbf{w}'\| \gg 1$ and

$$S \subset \{\mathbf{t} \in B : \big\|(\mathbf{w} - \mathbf{w}') \cdot L(\mathbf{t})\big\| \le \varepsilon\}.$$

Since $\|\mathbf{w} - \mathbf{w}'\| \gg 1$, the measure of the set on the right side of the above is $\ll \varepsilon$, where the implied constant depends on L and q.

The proof is complete. □

Given a compactly supported probability measure μ on $\mathbb{R}^d$, we let

$$\mathcal{E}_\alpha(\mu) = \iint \frac{\mathrm{d}\mu(X)\,\mathrm{d}\mu(X')}{\|X - X'\|^\alpha}$$

denote the α-dimensional energy of μ.

2.2. **Lemma.** *Let $0 < \alpha < 1$. Let μ be a probability measure supported on $B_{\mathbb{R}^n}(0,1)$ which satisfies*

$$\mathcal{E}_\alpha(\mu) \le C$$

for some $C \ge 1$. Then for every $R > 0$, we have

$$\left|\{\mathbf{t} \in \mathbb{R}^{n-2} : 1 \le \|\mathbf{t}\| \le 2, \mathcal{E}_\alpha(f_{\mathbf{t}}\mu) > R\}\right| \le C'/R$$

where $C' \ll \frac{C}{1-\alpha}$. In particular, $\mathcal{E}_\alpha(f_{\mathbf{t}}\mu) < \infty$ for (Lebesgue) a.e. $\mathbf{t}$.

Proof. We recall the standard argument which is based on Lemma 2.1.

Put $\mu_{\mathbf{t}} = f_{\mathbf{t}}\mu$. Using the definition of α-dimensional energy and the Fubini's theorem, we have

$$\begin{aligned}\int_B \mathcal{E}_\alpha(\mu_{\mathbf{t}})\,\mathrm{d}\mathbf{t} &= \int_B \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\mathrm{d}\mu(X)\,\mathrm{d}\mu(X')}{\|f_{\mathbf{t}}(X) - f_{\mathbf{t}}(X')\|^\alpha}\,\mathrm{d}\mathbf{t}\\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_B \frac{\mathrm{d}\mathbf{t}}{\|f_{\mathbf{t}}(X) - f_{\mathbf{t}}(X')\|^\alpha}\,\mathrm{d}\mu(X)\,\mathrm{d}\mu(X')\end{aligned}$$

Renormalizing with the factor $\|X - X'\|^\alpha$, we conclude that

$$\int_B \mathcal{E}_\alpha(\mu_{\mathbf{t}})\,\mathrm{d}\mathbf{t} = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_B \frac{\mathrm{d}\mathbf{t}}{\left\|\frac{f_{\mathbf{t}}(X)-f_{\mathbf{t}}(X')}{\|X-X'\|}\right\|^\alpha}\frac{\mathrm{d}\mu(X)\,\mathrm{d}\mu(X')}{\|X - X'\|^\alpha}. \tag{2.2}$$

Since $\alpha < 1$, applying Lemma 2.1, we conclude that

$$\int_B \left\|\frac{f_{\mathbf{t}}(X) - f_{\mathbf{t}}(X')}{\|X - X'\|}\right\|^{-\alpha}\mathrm{d}\mathbf{t} \ll \frac{1}{1-\alpha}.$$

This, (2.2), and $\mathcal{E}_\alpha(\mu) \le C$ imply that

$$\int_B \mathcal{E}_\alpha(\mu_{\mathbf{t}})\,\mathrm{d}\mathbf{t} \ll \frac{\mathcal{E}_\alpha(\mu)}{1-\alpha} \ll \frac{C}{1-\alpha}.$$

The claim in the lemma follows from this and the Chebyshev's inequality. □

Proof of Theorem 1.1. Let $s \in \mathbb{R}$ and $\mathbf{t} \in \mathbb{R}^{n-2}$. Then

$$\begin{aligned}\pi_{s\mathbf{t}}(r_1, \mathbf{w}, r_2) &= r_1 + \mathbf{w}\cdot L(s\mathbf{t}) + r_2 q(s\mathbf{t})\\ &= (1, s, s^2)\cdot f_{\mathbf{t}}(r_1, \mathbf{w}, r_2).\end{aligned} \tag{2.3}$$

Let $K \subset B_{\mathbb{R}^n}(0,1)$ be a compact subset, and let $\kappa = \min(1, \dim K)$. Let $0 < \alpha < \kappa$. By Frostman's lemma, there exists a probability measure μ supported on K and satisfying the following

$$\mu(B(X,\delta)) \le \delta^\alpha \quad \text{for all } X \in K.$$

Then by Lemma 2.2, applied with μ, there exists a conull subset $\Xi_\alpha \subset \mathbb{R}^{n-2}$ so that $\dim(f_{\mathbf{t}}(K)) \geq \alpha$ for all $\mathbf{t} \in \Xi_\alpha$. Applying this with $\alpha_n = \kappa - \frac{1}{n}$ for all $n \in \mathbb{N}$, we obtain a conull subset $\Xi \subset \mathbb{R}^{n-2}$ so that

$$\dim(f_{\mathbf{t}}(K)) \geq \kappa, \quad \text{for all } \mathbf{t} \in \Xi.$$

Let $\mathbf{t} \in \Xi$, and set $K_{\mathbf{t}} = f_{\mathbf{t}}(K)$. Then by [PYZ22, Thm. 1.3], see also [GGW22], for a.e. $s \in \mathbb{R}$, we have

$$\left\{x_1 + x_2 s + x_3 s^2 : (x_1, x_2, x_3) \in K_{\mathbf{t}}\right\} \subset \mathbb{R}$$

has dimension κ. This and (2.3) complete the proof. □

3. Proof of Theorem 1.2

We now turn to the proof of Theorem 1.2, the argument is a discretized version of the argument in §2 as we now explicate.

Let $F \subset \mathbb{R}^n$ be a finite, and let μ be the uniform measure on F. Our standing assumption is that for some $0 < \alpha \leq 1$ and some $C \geq 1$, we have

$$\mu(B_{\mathbb{R}^n}(X, \delta)) \leq C\delta^\alpha \qquad \text{for all } X \in F \text{ and all } \delta \geq \delta_0. \tag{3.1}$$

Without loss of generality, we will assume $\delta_0 = 2^{-k_0}$ for some $k_0 \in \mathbb{N}$; we will also assume that $\operatorname{diam}(F) \leq 1$.

For a finitely supported probability measure ρ on $\mathbb{R}^d$, define

$$\mathcal{E}^+_{\alpha,\rho} : \mathbb{R}^d \to \mathbb{R} \quad \text{by} \quad \hat{\mathcal{E}}_{\alpha,\rho}(X) = \int \|X - X'\|_+^{-\alpha} \, \mathrm{d}\rho(X')$$

where $\|X - X'\|_+ = \max\big(\|X - X'\|, \delta_0\big)$ for all $X, X' \in \mathbb{R}^d$ — this definition is motivated by the fact that we are only concerned with scales $\geq \delta_0$,

We recall the following standard lemma.

3.1. **Lemma.** *Let ρ be a finitely supported probability measure on $\mathbb{R}^d$. Assume that for some $X \in \mathbb{R}^d$ we have*

$$\hat{\mathcal{E}}_{\alpha,\rho}(X) \leq R.$$

Then for all $\delta \geq \delta_0$, we have

$$\rho(B_{\mathbb{R}^d}(X, \delta)) \leq R\delta^\alpha.$$

Proof. We include the proof for completeness. Let $\delta \geq \delta_0$, then

$$\begin{aligned}\delta^{-\alpha}\rho(B_{\mathbb{R}^d}(X, \delta)) &\leq \int_{B_{\mathbb{R}^d}(X,\delta)} \|X - X'\|_+^{-\alpha} \, \mathrm{d}\rho(X') \\ &\leq \int \|X - X'\|_+^{-\alpha} \, \mathrm{d}\rho(X') = \hat{\mathcal{E}}_{\alpha,\rho}(X) \leq R,\end{aligned}$$

as it was claimed. □

Recall our notation $B = \{\mathbf{t} \in \mathbb{R}^{n-2} : 1 \leq \mathbf{t} \leq 2\}$. For every $\mathbf{t} \in B$, let $\mu_{\mathbf{t}} = f_{\mathbf{t}}\mu$ where $f_{\mathbf{t}} : \mathbb{R}^n \to \mathbb{R}$ is defined as in (2.1):

$$f(r_1, \mathbf{w}, r_2) = \big(r_1, \mathbf{w} \cdot L(\mathbf{t}), r_2 q(\mathbf{t})\big), \tag{3.2}$$

and $\mathbb{R}^n = \{(r_1, \mathbf{w}, r_2) : r_i \in \mathbb{R}, \mathbf{w} \in \mathbb{R}^{n-2}\}$.

3.2. **Lemma.** *For every $X \in F$,*

$$\int_B \hat{\mathcal{E}}_{\alpha,\mu_{\mathbf{t}}}(f_{\mathbf{t}}X)\,\mathrm{d}\mathbf{t} \ll C\,|\log_2(\delta_0)|$$

where the implied constant is absolute.

Proof. Let $X \in F$. By the definitions, we have

$$\int_B \hat{\mathcal{E}}_{\alpha,\mu_{\mathbf{t}}}(f_{\mathbf{t}}X)\,\mathrm{d}\mathbf{t} = \int_B \int \left\| f_{\mathbf{t}}X - f_{\mathbf{t}}X' \right\|_+^{-\alpha} \mathrm{d}\mu(X')\,\mathrm{d}\mathbf{t}.$$

Renormalizing with $\|X - X'\|_+^{-\alpha}$ and using Fubini's theorem, we have

$$\int_B \hat{\mathcal{E}}_{\alpha,\mu_{\mathbf{t}}}(f_{\mathbf{t}}X)\,\mathrm{d}\mathbf{t} = \int \int_B \frac{1}{\frac{\|f_{\mathbf{t}}X - f_{\mathbf{t}}X'\|_+^\alpha}{\|X-X'\|_+^\alpha}}\,\mathrm{d}\mathbf{t}\,\left\|X - X'\right\|_+^{-\alpha} \mathrm{d}\mu(X').$$

For every $0 \leq k \leq k_0 - 1$, let

$$F_k(X) = \{X' \in F : 2^{-k-1} < \left\|X - X'\right\| \leq 2^{-k}\},$$

and let $F_{k_0}(X) = \{X' \in F : \|X - X'\| \leq 2^{-k_0}\}$.

For all $X' \in F_{k_0}$, we have $1 \leq \frac{\|f_{\mathbf{t}}X - f_{\mathbf{t}}X'\|_+}{\|X-X'\|_+} \leq 10$. Thus,

$$\int_{F_{k_0}} \int_B \frac{1}{\frac{\|f_{\mathbf{t}}X - f_{\mathbf{t}}X'\|_+^\alpha}{\|X-X'\|_+^\alpha}}\,\mathrm{d}\mathbf{t}\,\left\|X - X'\right\|_+^{-\alpha} \mathrm{d}\mu(X') \ll \mu(F_{k_0})2^{k_0\alpha} \ll C. \tag{3.3}$$

We now turn to the contribution of F_k to the above integral for $k < k_0$. If $X' \in F_k$, for some $k < k_0$, then $\|X - X'\|_+ = \|X - X'\|$ and we have

$$\int_{F_k} \int_B \frac{1}{\frac{\|f_{\mathbf{t}}X - f_{\mathbf{t}}X'\|_+^\alpha}{\|X-X'\|_+^\alpha}}\,\mathrm{d}\mathbf{t}\,\left\|X - X'\right\|_+^{-\alpha} \mathrm{d}\mu(X') = \int_{F_k} \int_B \frac{1}{\frac{\|f_{\mathbf{t}}X - f_{\mathbf{t}}X'\|_+^\alpha}{\|X-X'\|^\alpha}}\,\mathrm{d}\mathbf{t}\,\left\|X - X'\right\|^{-\alpha} \mathrm{d}\mu(X'). \tag{3.4}$$

By Lemma 2.1, we have

$$\int_B \frac{1}{\frac{\|f_{\mathbf{t}}X - f_{\mathbf{t}}X'\|_+^\alpha}{\|X-X'\|^\alpha}}\,\mathrm{d}\mathbf{t} \leq \int_B \frac{1}{\left\|\frac{f_{\mathbf{t}}X - f_{\mathbf{t}}X'}{\|X-X'\|}\right\|^\alpha}\,\mathrm{d}\mathbf{t} \ll 1$$

where the implied constant is absolute. Thus

$$(3.4) \ll \mu(F_k)2^{k\alpha} \ll C$$

This and (3.3) imply that

$$\int_B \hat{\mathcal{E}}_{\alpha,\mu_{\mathbf{t}}}(f_{\mathbf{t}}(X))\,\mathrm{d}\mathbf{t} \ll Ck_0 = C\,|\log_2(\delta_0)|$$

as we claimed. □

3.3. **Proposition.** *Let $0<\varepsilon<1$. There exists $B'\subset B$ with*

$$\left|B\setminus B'\right| \le \varepsilon^{-A'}\delta_0^{\varepsilon}$$

so that the following holds. For every $\mathbf{t}\in B'$, there exists $F_{\mathbf{t}}\subset F$ with

$$\mu(F\setminus F_{\mathbf{t}}) \le \varepsilon^{-A'}\delta_0^{\varepsilon}$$

so that for every $X\in F_{\mathbf{t}}$ and every $\delta\ge\delta_0$ we have

$$\mu_{\mathbf{t}}\big(B_{\mathbb{R}^3}(f_{\mathbf{t}}(X),\delta)\big) \ll \varepsilon^{-A'}\delta_0^{-3\varepsilon}\cdot\delta^{\alpha}$$

where A' is absolute and the implied constant depends on C.

Proof. The proof is based on Lemma 3.2 and Chebychev's inequality as we now explicate. First note that we may assume δ_0 is small enough (polynomially in ε) so that

$$\delta_0^{-\varepsilon/10} > |\log\delta_0|$$

otherwise the statement follows trivially.

By Lemma 3.2, we have

$$\int_B \hat{\mathcal{E}}_{\alpha,\mu_{\mathbf{t}}}(f_{\mathbf{t}}(X))\,\mathrm{d}\mathbf{t} \le C'\,|\log_2(\delta_0)|\,. \tag{3.5}$$

where $C'\ll C$. Averaging (3.5), with respect to μ, and using Fubini's theorem we have

$$\int_B\int \hat{\mathcal{E}}_{\alpha,\mu_{\mathbf{t}}}(f_{\mathbf{t}}(X))\,\mathrm{d}\mu(X)\,\mathrm{d}\mathbf{t} \le C'\,|\log_2(\delta_0)|\,. \tag{3.6}$$

Let $B'=\{\mathbf{t}\in B:\int\hat{\mathcal{E}}_{\alpha,\mu_{\mathbf{t}}}(f_{\mathbf{t}}(X))\,\mathrm{d}\mu(X)<C'\delta_0^{-2\varepsilon}\}$. Then by (3.6) and Chebychev's inequality we have

$$\mu(B\setminus B')\le\delta_0^{\varepsilon}.$$

Let now $\mathbf{t}\in B'$, then

$$\int \hat{\mathcal{E}}_{\alpha,\mu_{\mathbf{t}}}(f_{\mathbf{t}}(X))\,\mathrm{d}\mu(X) \le C'\delta_0^{-2\varepsilon}. \tag{3.7}$$

For every $\mathbf{t}\in B'$, set

$$F_{\mathbf{t}}=\{X\in F:\hat{\mathcal{E}}_{\alpha,\mu_{\mathbf{t}}}(f_{\mathbf{t}}(X))<C'\delta_0^{-3\varepsilon}\}$$

Then (3.7) and Chebychev's inequality again imply that $\mu(F\setminus F_{\mathbf{t}})\le\delta_0^{\varepsilon}$.

Altogether, for every $\mathbf{t}\in B'$ and $X\in F_{\mathbf{t}}$, we have

$$\hat{\mathcal{E}}_{\alpha,\mu_{\mathbf{t}}}(f_{\mathbf{t}}(X)) = \int \left\|f_{\mathbf{t}}(X)-f_{\mathbf{t}}(X')\right\|_+^{-\alpha}\mathrm{d}\mu_{\mathbf{t}}(X') \le C'\delta_0^{-3\varepsilon}$$

This and Lemma 3.1 imply that for every $\mathbf{t}\in B'$ and $X\in F_{\mathbf{t}}$, we have

$$\mu_{\mathbf{t}}\big(B_{\mathbb{R}^3}(f_{\mathbf{t}}(X),\delta)\big) \le C'\delta_0^{-3\varepsilon}\cdot\delta^{\alpha}\quad\text{for all }\delta\ge\delta_0.$$

The proof is complete. □

Proof of Theorem 1.2. We now turn to the proof of Theorem 1.2. As it was done in the proof of Theorem 1.1, we will use the following observation: for all $s \in \mathbb{R}$ and $\mathbf{t} \in \mathbb{R}^{n-2}$, we have

$$\begin{aligned} \pi_{s\mathbf{t}}(r_1, \mathbf{w}, r_2) &= r_1 + \mathbf{w} \cdot L(s\mathbf{t}) + r_2 q(s\mathbf{t}) \\ &= (1, s, s^2) \cdot f_{\mathbf{t}}(r_1, \mathbf{w}, r_2). \end{aligned} \tag{3.8}$$

Apply Proposition 3.3 with ε as in the statement of Theorem 1.2. Let $B' \subset B$ be as in that proposition, and for every $\mathbf{t} \in B'$, let $F_{\mathbf{t}}$ be as in that proposition. Then we have

$$\mu_{\mathbf{t}}\big(B_{\mathbb{R}^3}(f_{\mathbf{t}}(X), \delta)\big) \le C'\varepsilon^{-A'}\delta_0^{-3\varepsilon} \cdot \delta^{\alpha} \quad \text{for all } X \in F_{\mathbf{t}} \text{ and } \delta \ge \delta_0. \tag{3.9}$$

Let $K_{\mathbf{t}} = f_{\mathbf{t}}(F_{\mathbf{t}}) \subset \mathbb{R}^3$ and let $\rho_{\mathbf{t}}$ be the restriction of μ_t to $K_{\mathbf{t}}$ normalized to be a probability measure. Then (3.9) and the fact that $\mu(F \setminus F_{\mathbf{t}}) \le \delta_0^{\varepsilon}$ imply that

$$\rho_{\mathbf{t}}\big(B_{\mathbb{R}^3}(Y, \delta)\big) \le 2C'\varepsilon^{-A'}\delta_0^{-3\varepsilon} \cdot \delta^{\alpha} \quad \text{for all } Y \in K_{\mathbf{t}} \text{ and } \delta \ge \delta_0. \tag{3.10}$$

This in particular implies that $K_{\mathbf{t}}$ and $\rho_{\mathbf{t}}$ satisfy the conditions in [LM21, Thm. B], see also [PYZ22] and [GGW22, Thm. 2.1]. Apply [LM21, Thm. B] with ε; thus, there there exists $J_{\delta,\mathbf{t}} \subset [0, 2]$ with

$$|[0, 2] \setminus J_{\delta,\mathbf{t}}| \le \hat{C}\varepsilon^{-D}\delta^{\varepsilon}$$

and for all $s \in J_{\delta,\mathbf{t}}$ there is a subset $K_{\delta,\mathbf{t},s} \subset K_{\mathbf{t}}$ with

$$\rho_{\mathbf{t}}(K_{\mathbf{t}} \setminus K_{\delta,\mathbf{t},s}) \le \hat{C}\varepsilon^{-D}\delta^{\varepsilon}$$

so that for all $Y \in K_{\delta,\mathbf{t},s}$, we have

$$\rho_{\mathbf{t}}\big(\{Y' \in K_{\mathbf{t}} : \big|(1, s, s^2) \cdot (Y - Y')\big| \le \delta\}\big) \le \hat{C}\varepsilon^{-D}\delta_0^{-3\varepsilon} \cdot \delta^{\alpha - 7\varepsilon};$$

Let $A = \max\{A', D\}$. In view of the definition of $\rho_{\mathbf{t}}$ and (3.8), we have the following. For every $\mathbf{t} \in B'$ and $s \in J_{\delta,\mathbf{t}}$, put $F_{\delta,s\mathbf{t}} = F \cap f_{\mathbf{t}}^{-1}(K_{\delta,\mathbf{t},s})$. Then

$$\#(F \setminus F_{\delta,s\mathbf{t}}) \le 10\hat{C}\varepsilon^{-A}\delta^{\varepsilon} \cdot (\#F),$$

and for every $X \in F_{\delta,s\mathbf{t}}$, we have

$$\#\{X' \in F_{\delta,s\mathbf{t}} : \big|\pi_{s\mathbf{t}}(X) - \pi_{s\mathbf{t}}(X')\big| \le \delta\}\big) \le \hat{C}\varepsilon^{-A}\delta_0^{-3\varepsilon}\delta^{\alpha - 7\varepsilon}. \tag{3.11}$$

This finishes the proof. Indeed, let $B_\delta \subset B$ be the set of $\mathbf{t} \in B$ for which there exists $F_{\delta,\mathbf{t}} \subset F$ with

$$\#(F \setminus F_{\delta,\mathbf{t}}) \le 100\hat{C}\varepsilon^{-D}\delta^{\varepsilon} \cdot (\#F)$$

so that for all $X \in F_{\delta,\mathbf{t}}$ we have

$$\#\{X' \in F_{\delta,\mathbf{t}} : \big|\pi_{\mathbf{t}}(X) - \pi_{\mathbf{t}}(X')\big| \le \delta\}\big) \le \hat{C}\varepsilon^{-A}\delta_0^{-10\varepsilon}\delta^{\alpha}.$$

Then (3.11) implies that for every $\mathbf{t}' \in B'$ and $s \in J_{\delta,\mathbf{t}'}$, we have $s\mathbf{t}' \in B_\delta$, so long as $s\mathbf{t}' \in B$. In particular, we conclude that

$$|B \setminus B_\delta| \ll \varepsilon^{-A}\delta^{\varepsilon}$$

as we aimed to prove. □

4. Theorem 1.2 and the isometry group of $\mathbb{H}^n$

Let $n \geq 3$, and let

$$Q_0(x_1, x_1, \ldots, x_{n+1}) = 2x_1 x_{n+1} - \sum_{i=1}^{n} x_i^2$$

Then $G = \mathrm{SO}(Q_0)^\circ \simeq \mathrm{SO}(n,1)^\circ$ is the group of orientation preserving isometries of $\mathbb{H}^n$. Let

$$\mathrm{Lie}(G) = \{A \in \mathfrak{sl}_{n+1}(\mathbb{R}) : A^T Q_0 + Q_0 A = 0\}$$

where we identify Q_0 and the corresponding symmetric matrix, i.e.,

$$Q_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -I_{n-2} & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Let $H \subset G$ be the stabilizer of $e_n = (0, \ldots, 1, 0)$. Then

$$\mathrm{Lie}(G) = \mathrm{Lie}(H) \oplus \mathfrak{r}$$

where $\mathfrak{r}$ is invariant under conjugation by H and $\dim \mathfrak{r} = n$. More explicitly,

$$\mathfrak{r} = \left\{X(r_1, \mathbf{w}, r_2) : r_1, r_2 \in \mathbb{R}, \mathbf{w} \in \mathbb{R}^{n-2}\right\}$$

where for $r_1, r_2 \in \mathbb{R}$ and $\mathbf{w} \in \mathbb{R}^{n-2}$,

$$X(r_1, \mathbf{w}, r_2) = \begin{pmatrix} 0 & \mathbf{0} \quad r_1 & 0 \\ \mathbf{0}^T & & \mathbf{0}^T \\ & R(\mathbf{w}) & \\ -r_2 & & -r_1 \\ 0 & \mathbf{0} \quad r_2 & 0 \end{pmatrix} \in \mathrm{Mat}_{n+1}(\mathbb{R}) \tag{4.1}$$

here $\mathbf{0} \in \mathbb{R}^{n-2}$ and $R(\mathbf{w}) = \begin{pmatrix} \mathbf{0}_{n-2} & \mathbf{w}^T \\ -\mathbf{w} & 0 \end{pmatrix} \in \mathrm{Mat}_{n-1}(\mathbb{R})$.

We may identify $\mathfrak{r}$ with $\mathbb{R}^n$ using the above coordinates. With this notation, put

$$\mathfrak{r}^+ = \{X(r_1, 0, 0) : r_1 \in \mathbb{R}\} \simeq \mathbb{R}.$$

Define the subgroup $U \subset H$ as follows. For every $\mathbf{t} \in \mathbb{R}^{n-2}$, let

$$u_{\mathbf{t}} = \begin{pmatrix} 1 & \mathbf{t} \quad 0 & \frac{1}{2}\|\mathbf{t}\|^2 \\ & & \mathbf{t}^T \\ \mathbf{0}^T & I_{n-2} & \\ & & 0 \\ 0 & \mathbf{0} & 1 \end{pmatrix} \in \mathrm{Mat}_{n+1}(\mathbb{R}).$$

where $\mathbf{0} \in \mathbb{R}^{n-1}$. Put

$$U = \{u_{\mathbf{t}} : \mathbf{t} \in \mathbb{R}^{n-2}\}.$$

It is worth noting that if a_s denotes the one parameter diagonal subgroup of H defined by

$$a_s e_1 = e^s e_1, \quad a_s e_{n+1} = e^{-s} e_{n+1}, \quad a_s e_i = e_i \quad 2 \leq i \leq n,$$

then

$$U = \{h \in H : \lim_{s\to-\infty} a_s h a_{-s} = 1\} \quad \text{and}$$

$$\mathfrak{r}^+ = \{X \in \mathfrak{r} : \lim_{s\to-\infty} a_s X a_{-s} = 0\}.$$

For every $\mathbf{t} \in \mathbb{R}^{n-2}$, define

$$\xi_{\mathbf{t}} : \mathfrak{r} \to \mathfrak{r}^+ \quad \text{by} \quad \xi_{\mathbf{t}}(X) = (u_{\mathbf{t}} X u_{-\mathbf{t}})^+.$$

where for $X \in \mathfrak{r}$, X^+ denote the orthogonal projection to $\mathfrak{r}^+$.

We have the following lemma.

4.1. **Lemma.** *Identify $\mathfrak{r}^+$ and $\mathbb{R}$ as above. Then*

$$\xi_{\mathbf{t}}(X(r_1, \mathbf{w}, r_2)) = \pi_{\mathrm{id},q_{\mathrm{st}},\mathbf{t}}(r_1, \mathbf{w}, r_2)$$

where $\mathrm{id} : \mathbb{R}^{n-2} \to \mathbb{R}^{n-2}$ *is the identity map and* $q_{\mathrm{st}}(\mathbf{t}) = \frac{1}{2}\|\mathbf{t}\|^2$.

Proof. The proof is based on a direct computation as we now explicate.

To simplify the notation slightly, we will write $\tilde{\mathbf{t}} = (\mathbf{t}, 0)$ and $\tilde{r}_i = (\mathbf{0}, r_i)$. Note that $\tilde{\mathbf{t}} \cdot \tilde{r}_i = 0$. Recall also that

$$\xi_{\mathbf{t}}(X) = (u_{\mathbf{t}} X(r_1, r_2, \mathbf{w}) u_{-\mathbf{t}})^+$$

We have

$$u_{\mathbf{t}} X(r_1, r_2, \mathbf{w}) = \begin{pmatrix} 1 & \tilde{\mathbf{t}} & \frac{1}{2}\|\mathbf{t}\|^2 \\ \mathbf{0}^T & I_{n-2} & \tilde{\mathbf{t}}^T \\ 0 & \mathbf{0} & 1 \end{pmatrix} \begin{pmatrix} 0 & \tilde{r}_1 & 0 \\ -\tilde{r}_2^T & R(\mathbf{w}) & -\tilde{r}_1 \\ 0 & \tilde{r}_2 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & \tilde{r}_1 + \tilde{\mathbf{t}} R(\mathbf{w}) + \frac{1}{2}\|\mathbf{t}\|^2 \tilde{r}_2 & 0 \\ -\tilde{r}_2^T & R(\mathbf{w}) + \tilde{\mathbf{t}}^T \tilde{r}_2 & -\tilde{r}_1 \\ 0 & \tilde{r}_2 & 0 \end{pmatrix}$$

Multiplying this with $u_{-\mathbf{t}}$, we have

$$u_{\mathbf{t}} X(r_1, r_2, \mathbf{w}) u_{-\mathbf{t}} =$$

$$\begin{pmatrix} 0 & \tilde{r}_1 + \tilde{\mathbf{t}} R(\mathbf{w}) + \frac{1}{2}\|\mathbf{t}\|^2 \tilde{r}_2 & 0 \\ -\tilde{r}_2^T & R(\mathbf{w}) + \tilde{\mathbf{t}}^T \tilde{r}_2 & -\tilde{r}_1 \\ 0 & \tilde{r}_2 & 0 \end{pmatrix} \begin{pmatrix} 1 & -\tilde{\mathbf{t}} & \frac{1}{2}\|\mathbf{t}\|^2 \\ \mathbf{0}^T & I_{n-2} & -\tilde{\mathbf{t}}^T \\ 0 & \mathbf{0} & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & \tilde{r}_1 + \tilde{\mathbf{t}} R(\mathbf{w}) + \frac{1}{2}\|\mathbf{t}\|^2 \tilde{r}_2 & * \\ * & * & * \\ * & * & * \end{pmatrix}$$

Since $\tilde{\mathbf{t}} R(\mathbf{w}) = (\mathbf{0}, \mathbf{t} \cdot \mathbf{w})$, the above and the definitions of $\xi_{\mathbf{t}}$ and $\pi_{\mathbf{t}}$ imply

$$\xi_{\mathbf{t}}(X) = r_1 + \mathbf{t} \cdot \mathbf{w} + \frac{1}{2}\|\mathbf{t}\|^2 r_2 = \pi_{\mathrm{id},q_{\mathrm{st}},\mathbf{t}}(r_1, \mathbf{w}, r_2)$$

as we claimed. $\square$

In view of Lemma 4.1, the following theorem is a restatement of Theorem 1.2 in the language of adjoint action of U on $\mathfrak{r}$.

4.2. **Theorem.** *Let $0 < \alpha \leq 1$, and let $0 < \delta_0 \leq 1$. Let $F \subset B_{\mathfrak{r}}(0,1)$ be a finite set satisfying the following*

$$\#(B_{\mathfrak{r}}(X,\delta) \cap F) \leq C\delta^{\alpha} \cdot (\#F) \qquad \text{for all } X \in F \text{ and all } \delta \geq \delta_0$$

where $C \geq 1$.

Let $0 < \varepsilon < \alpha/100$. For every $\delta \geq \delta_0$, there exists a subset $B_\delta \subset B := \{\mathbf{t} \in \mathbb{R}^{n-2} : 1 \leq \|\mathbf{t}\| \leq 2\}$ with

$$|B \setminus B_\delta| \ll \varepsilon^{-A}\delta^{\varepsilon}$$

so that the following holds. Let $\mathbf{t} \in B_\delta$, there exists $F_{\delta,\mathbf{t}} \subset F$ with

$$\#(F \setminus F_{\delta,\mathbf{t}}) \ll \varepsilon^{-A}\delta^{\varepsilon} \cdot (\#F)$$

such that for all $X \in F_{\delta,\mathbf{t}}$, we have

$$\#\big(\{X' \in F_{\delta,\mathbf{t}} : |\xi_{\mathbf{t}}(X') - \xi_{\mathbf{t}}(X)| \leq \delta\}\big) \leq C\delta_0^{-10\varepsilon} \cdot \delta^{\alpha} \cdot (\#F)$$

where A and the implied constants are absolute.

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