An $N \times N$ input Queued Crossbar Switch Operating under a Maximum Weight Matching Policy in Heavy Traffic

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Outline

- $N \times N$ input queued packet switch: example of a stochastic processing network with simultaneous "service" of more than one buffer (cf. joining)

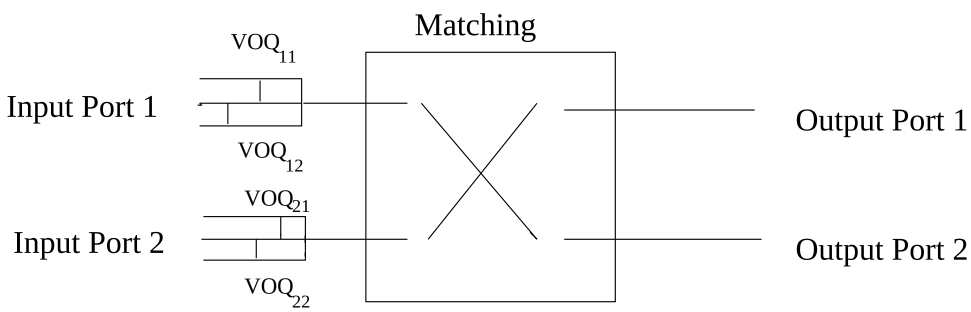
- α-max weight matching policy: solution of an optimization problem

- Stability: Lyapunov function and fluid model

- Heavy traffic diffusion approximation for workload: state space collapse, invariance principle, SRBM ($\alpha = 1$)

- On-going research ($\alpha \neq 1$)

$N \times N$ Input Queued Crossbar Switch



A 2 by 2 Input Queued Switch

- N input ports and N output ports

- Time is slotted (discrete)

- Packets buffered in virtual output queues ($VOQ_{11}, \ldots, VOQ_{1N}, \ldots, VOQ_{N1}, \ldots, VOQ_{NN}$)

- In each time slot, at most one packet can arrive to each input port, at most one packet can be transferred from each input port, and at most one packet can be transferred to each output port

- Scheduling control: need to choose a matching of input ports to output ports

α-Maximum Weight Matching Algorithm ($\alpha \in (0, \infty)$)

A *matching* is an $N \times N$ permutation matrix $\pi = (\pi_{ij})$ (there are $N!$ possible matchings)

Given $Q_{ij}(n)$, the number of packets in VOQ_{ij} at time n, the associated weight of a matching π is

$$\omega_\pi(Q(n)) = \sum_{i=1}^{N} \sum_{j=1}^{N} \pi_{ij}(Q_{ij}(n))^\alpha$$

Choose a maximum weight matching at time n whose effect is felt at time $n+1$:

$$\pi^*(n) \equiv \text{argmax}_\pi \{\omega_\pi(Q(n))\}$$

Special case: $\alpha = 1$ — maximum weight matching

Dynamic Equations

$E_{ij}(n)$ = the number of packets arrived to VOQ_{ij} in time $(0, n]$

$T_\pi(n)$ = the number of time slots that the matching π has taken effect in $(0, n]$

$D_{ij}(n)$ = the number of packets departed from VOQ_{ij} in time $(0, n]$

$$\begin{aligned}
Q_{ij}(n) &= Q_{ij}(0) + E_{ij}(n) - D_{ij}(n) \\
D_{ij}(n) &= \sum_\pi \sum_{l=1}^{n} \pi_{ij} 1_{\{Q_{ij}(l-1)>0\}} (T_\pi(l) - T_\pi(l-1)) \\
T_\pi(\cdot) & \quad \text{is non-decreasing and } \sum_\pi T_\pi(n) = n
\end{aligned}$$

$$T_\pi(n) - T_\pi(n-1) = 0 \text{ if } \exists\ \sigma :\ \omega_\sigma(n-1) > \omega_\pi(n-1)$$

Extend the definitions of E_{ij}, D_{ij}, T_π, Q_{ij} to the time interval $[0, \infty)$ in a piecewise constant manner.

Stability

Assume there exist $\lambda_{ij} > 0$ such that a.s.,

$$\lambda_{ij} = \lim_{n\to\infty} \frac{E_{ij}(n)}{n}, \quad i, j = 1, \ldots, N$$

Nominal Stability Condition (SC)

$$\begin{aligned} \sum_{i=1}^{N} \lambda_{ij} &< 1, \qquad \text{for } j = 1, \ldots, N \\ \sum_{j=1}^{N} \lambda_{ij} &< 1, \qquad \text{for } i = 1, \ldots, N \end{aligned}$$

Theorem If (SC) holds, then the switch is "stable" under the α-maximum weight matching algorithm.

($\alpha = 1$: McKeown et al. (1996), i.i.d. Bernoulli arrivals; Tassiulas and Ephremides (1992), similar results for radio networks; Dai and Prabhakar (2000), general arrivals; $\alpha \in (0, \infty)$: Keslassy and McKeown (2001), i.i.d. Bernoulli arrivals; Shah (2001), general arrivals)

Heavy Traffic

Assume

$$\begin{aligned}\sum_{i=1}^{N} \lambda_{ij} &= 1, \quad j = 1, \ldots, N \\ \sum_{j=1}^{N} \lambda_{ij} &= 1, \quad i = 1, \ldots, N\end{aligned}$$

Lemma (Birkhoff-von Neumann)
The doubly stochastic matrix λ can be expressed as a convex combination of matchings π (permutation matrices)

(cf. Stolyar (2004): other extreme — only one input port or one output port heavily loaded)

Workload Process

Define $(2N-1)$-dimensional workload W:

$$
\begin{aligned}
W_i &= \sum_{j=1}^{N} Q_{ij}, \quad i = 1, \ldots, N-1, \\
W_{N-1+j} &= \sum_{i=1}^{N} Q_{ij}, \quad j = 1, \ldots, N-1, \\
W_{2N-1} &= \sum_{i,j=1}^{N} Q_{ij}
\end{aligned}
$$

Write $W = AQ$

Will also need the (symmetric) workload S:

$$
\begin{aligned}
S_i &= \sum_{j=1}^{N} Q_{ij}, \quad i = 1, \ldots, N, \\
S_{N+j} &= \sum_{i=1}^{N} Q_{ij}, \quad j = 1, \ldots, N,
\end{aligned}
$$

Write $S = BQ$ and $S = CW$.

Fluid Model

(functional law of large numbers approximation)

For $i, j = 1, \ldots, N$, $t \geq 0$,

$$\bar{Q}_{ij}(t) \;=\; \bar{Q}_{ij}(0) + \lambda_{ij} t - \bar{D}_{ij}(t) \geq 0$$

$$\sum_{\pi} \bar{T}_{\pi}(t) \;=\; t,\ \bar{T}_{\pi}(\cdot) \uparrow,\ \bar{T}_{\pi}(0) = 0$$

$\bar{Q}, \bar{D}, \{\bar{T}_{\pi}\}$ are absolutely continuous. A time t is regular if $\bar{Q}, \bar{D}, \{\bar{T}_{\pi}\}$ are differentiable at t (occurs for a.e. t)

At a regular time t,

$$\frac{d}{dt}\bar{D}_{ij}(t) \;=\; \begin{cases} \sum_{\pi} \pi_{ij} \frac{d}{dt}\bar{T}_{\pi}(t) & \text{if } \bar{Q}_{ij}(t) > 0 \\ \lambda_{ij} & \text{if } \bar{Q}_{ij}(t) = 0 \end{cases}$$

$$\frac{d}{dt}\bar{T}_{\pi}(t) = 0 \text{ if } \exists\ \sigma :\ \omega_{\sigma}(\bar{Q}(t)) > \omega_{\pi}(\bar{Q}(t))$$

Lyapunov Function

$$F(q) = \frac{1}{\alpha + 1} \sum_{i=1}^{N} \sum_{j=1}^{N} q_{ij}^{\alpha+1}$$

Given a fluid model solution, let

$$\begin{aligned} m(t) &= \max_{\pi} \omega_{\pi}(\bar{Q}(t)) = \max_{\pi} \left(\sum_{i,j=1}^{N} \pi_{ij} \bar{Q}_{ij}^{\alpha}(t) \right) \\ \Pi(t) &= \{\pi : \omega_{\pi}(\bar{Q}(t)) = m(t)\} \end{aligned}$$

At a regular point t,

$$\begin{aligned}
\frac{d}{dt}F(\bar{Q}(t)) &= \sum_{i,j=1}^{N} \bar{Q}_{ij}^{\alpha}(t) \left(\lambda_{ij} - \sum_{\pi} \pi_{ij} \frac{d}{dt}\bar{T}_{\pi}(t) \right) \\
&\leq m(t) - \sum_{\pi \in \Pi(t)} \frac{d}{dt}\bar{T}_{\pi}(t) \sum_{i,j=1}^{N} \pi_{ij} \bar{Q}_{ij}^{\alpha}(t) \\
&= m(t) - \sum_{\pi \in \Pi(t)} \frac{d}{dt}\bar{T}_{\pi}(t) m(t) \\
&= m(t) - m(t) = 0
\end{aligned}$$

Invariant States

$q \in \mathbb{R}_+^{N^2}$ is an invariant state for the fluid model if $\exists$ a fluid model solution $\bar{Q}(\cdot)$:

$$\bar{Q}(t) = q \text{ for all } t \geq 0.$$

Given $s \in \mathbb{R}_+^{2N}$, let $\Delta(s)$ denote the optimal solution of

$$\text{minimize} \quad F(q) \qquad \text{such that } Bq \geq s, \quad q \in \mathbb{R}_+^{N^2}$$

Theorem (Shah-Wischik)
q is an invariant state if and only if $q = \Delta(s(q))$ where $s(q) = Bq$.

Note: Each invariant state q is of the form $q = (B'p)^{1/\alpha}$ for $p \in \mathbb{R}_+^{2N}$ (via Lagrange multipliers)

(Multiplicative) State Space Collapse

For each $r > 0$, define diffusion scaled processes:

$$\begin{aligned} \hat{Q}^r(t) &= Q(r^2 t)/r \\ \hat{W}^r(t) &= A\hat{Q}^r(t) \end{aligned}$$

Assume the switch starts empty and packet arrivals are i.i.d. (with finite means and variances)

Theorem (Shah-Wischik)

For any $T \geq 0$,

$$\frac{\left\|\hat{Q}^r(\cdot) - \Delta(C\hat{W}^r(\cdot))\right\|_T}{\left\|\hat{Q}^r(\cdot)\right\|_T \vee 1} \to 0$$

in probability as $r \to \infty$.

(cf. Bramson, 1998 for multiclass HL queueing networks)

Diffusion Scaling in Heavy traffic

Define

$$Y_{ij}(t) = \int_0^{\lfloor t \rfloor} 1_{\{Q_{ij}(s-)=0\}} d\left(\sum_{\pi} \pi_{ij} T_{\pi}(s)\right)$$

Define centered, rescaled processes

$$\begin{aligned}
\hat{E}^r(t) &= (E(r^2 t) - \lambda r^2 t)/r \\
\hat{Q}^r(t) &= Q(r^2 t)/r \\
\hat{W}^r(t) &= A\hat{Q}^r(t) \\
\hat{Y}^r(t) &= Y(r^2 t)/r
\end{aligned}$$

Workload Process

$$\hat{W}^r(t) = \hat{X}^r(t) + A\hat{Y}^r(t)$$

where $\hat{X}^r(t) = A\hat{E}^r(t) + o(1)$, $t \geq 0$

Conjecture of Shah and Wischik

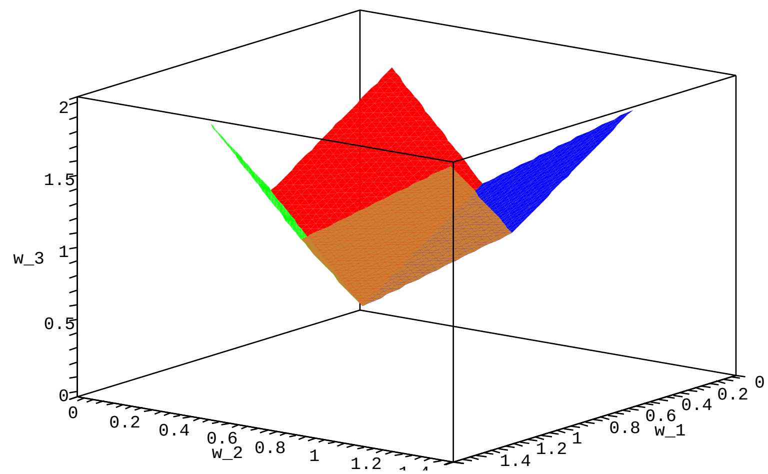
$$\hat{W}^r \Rightarrow \tilde{W} \text{ as } r \to \infty$$

where $\tilde{W} = \tilde{X} + A\tilde{Y}$ is a semimartingale reflecting Brownian motion living in a cone $\overline{G}^\alpha \subset \mathbb{R}_+^{2N-1}$ bounded by N^2 faces

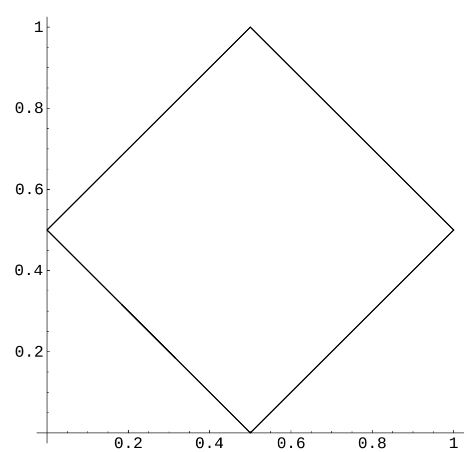
$$\begin{aligned} \partial G_{ij}^\alpha &= \{w \in \overline{G}^\alpha : w = A(B'p)^{1/\alpha},\ p \in \mathbb{R}_+^{2N}, \\ &\qquad p_i = p_{N+j} = 0\} \end{aligned}$$

The column of A indexed by ij is the direction of reflection γ^{ij} on the boundary face ∂G_{ij}^α where $\tilde{Y}_{ij}$ can increase

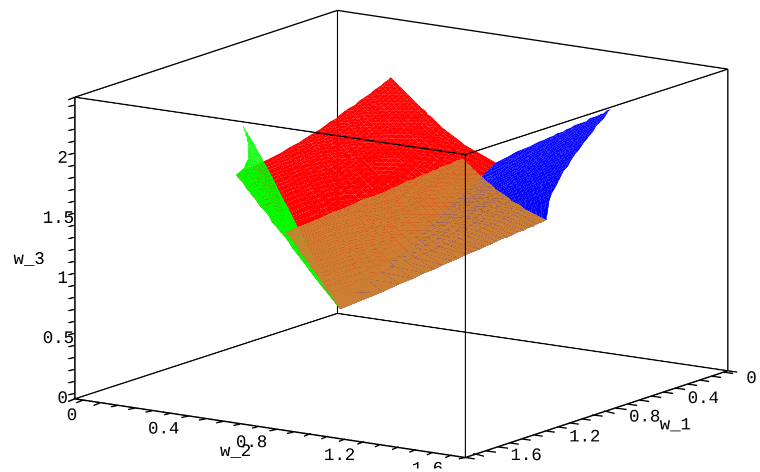
2×2 Switch: the Cone $\overline{G}^{\alpha}$



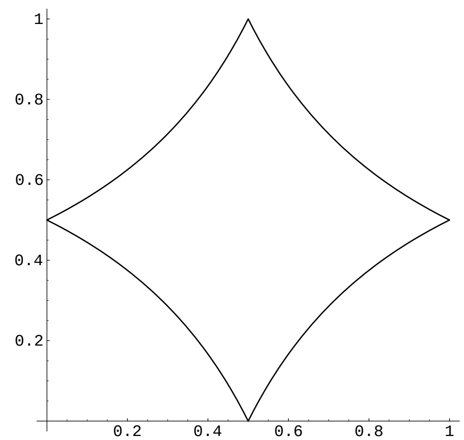
$\alpha = 1$



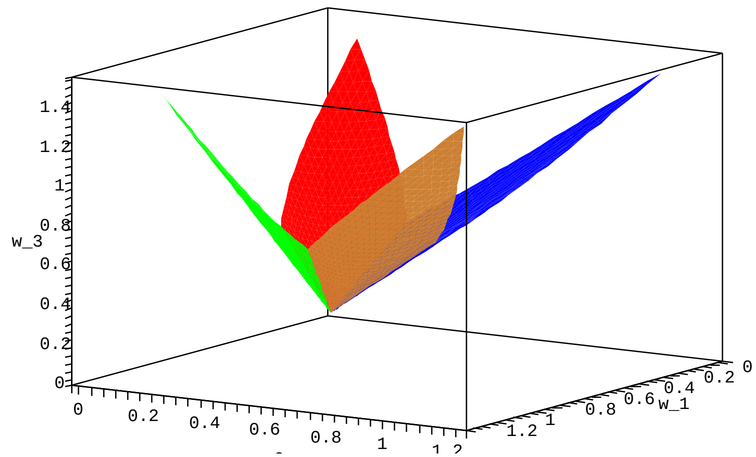
Cross-section: $\alpha = 1$



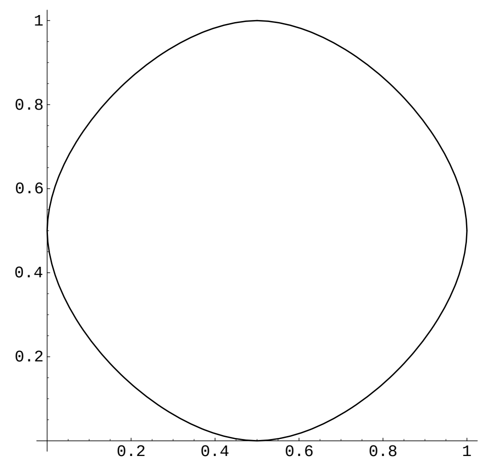
$\alpha = 2$



Cross-section: $\alpha = 2$



$\alpha = 0.5$



Cross-section: $\alpha = 0.5$

Diffusion Approximation in Heavy traffic

Theorem (Kang-W) Assume $\alpha = 1$.

$$(\hat{W}^r, \hat{Q}^r) \Rightarrow (\tilde{W}, \tilde{Q}) \text{ as } r \to \infty$$

where $\tilde{W} = \tilde{X} + A\tilde{Y}$ is an SRBM in $\overline{G}^1$ and $\tilde{Q} = \Delta(C\tilde{W})$.

Proof: Uses an invariance principle for domains with piecewise smooth boundaries.

Key Condition (cf. Dai-W 1995):

There is a constant $a \in (0,1)$, and functions $b(\cdot) = (b_{ij}(\cdot))$ and $c(\cdot) = (c_{ij}(\cdot))$ from ∂G^1 into $\mathbb{R}_+^{N^2}$ such that for each $x \in \partial G^1$,

(i) $\sum_{ij} b_{ij}(x) = 1$,

$$\min_{kl \in \mathcal{I}(x)} \left\langle \sum_{ij \in \mathcal{I}(x)} b_{ij}(x) n^{ij}(x), \gamma^{kl}(x) \right\rangle \geq a,$$

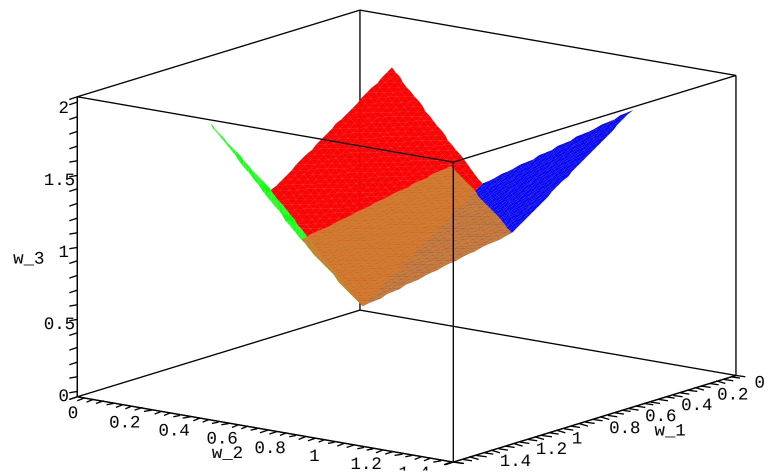
(ii) $\sum_{ij \in \mathcal{I}(x)} c_{ij}(x) = 1$,

$$\min_{kl \in \mathcal{I}(x)} \left\langle \sum_{ij \in \mathcal{I}(x)} c_{ij}(x) \gamma^{ij}(x), n^{kl}(x) \right\rangle \geq a.$$

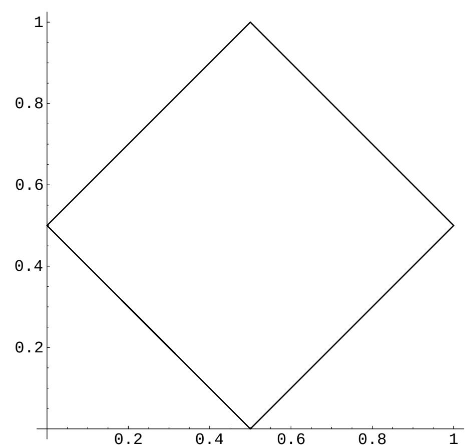
Here

$$I(x) = \{ij : x \in \partial G^1_{ij}\}$$

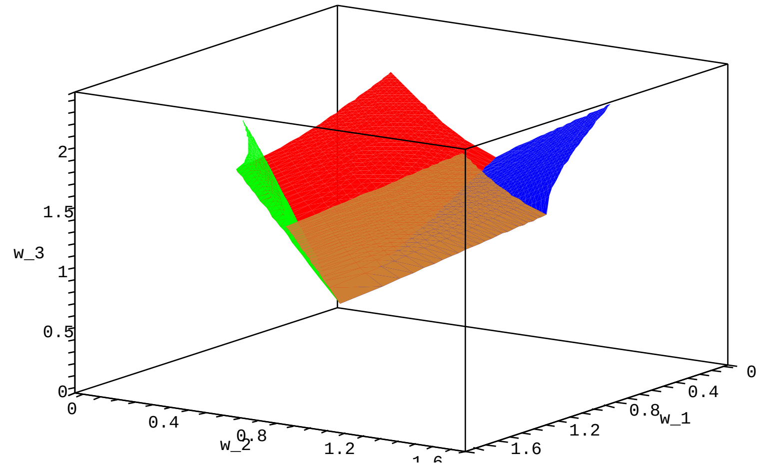
2×2 Switch: the Cone $\overline{G}^{\alpha}$



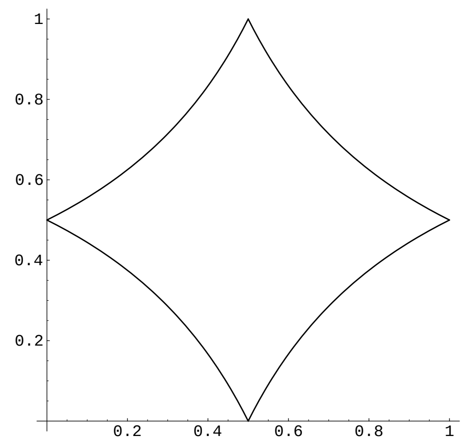
$\alpha = 1$



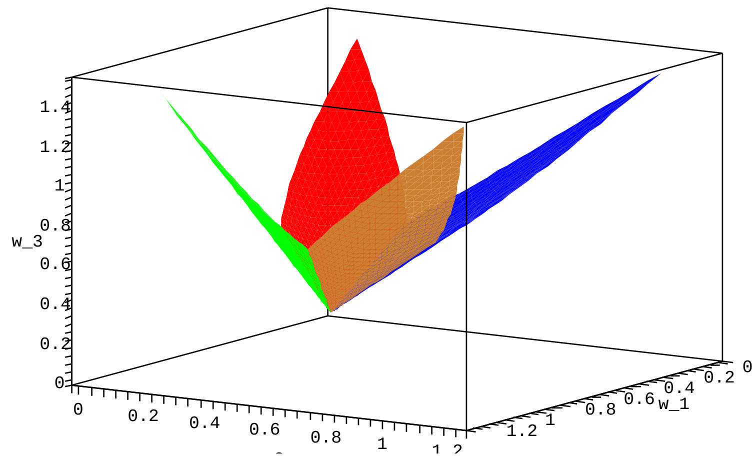
Cross-section: $\alpha = 1$



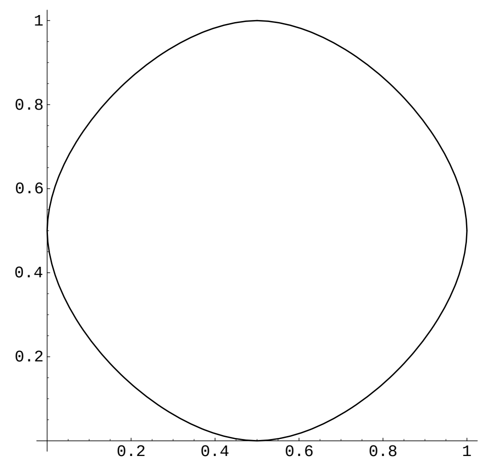
$\alpha = 2$



Cross-section: $\alpha = 2$



$\boldsymbol{\alpha = 0.5}$



Cross-section: $\boldsymbol{\alpha = 0.5}$