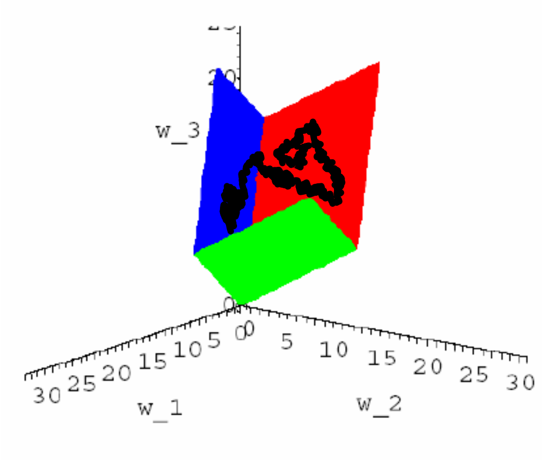
RBM and Queueing Theory: An Early History
Part 2. 1981-2000



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Based on my randonnée through RBM and related topics


Mike Harrison

RBM in a Wedge

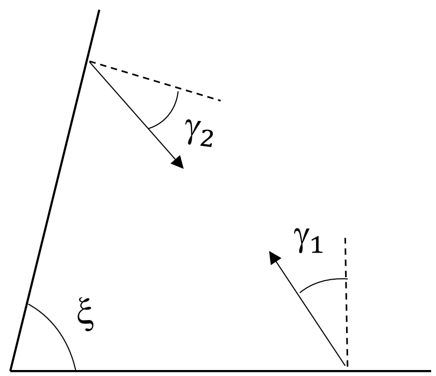

SRS Varadhan

- Mike Harrison's course on Stochastic Calculus -> Brownian Motion and Stochastic Flow Systems (Winter 1981)
- Follow-on discussions with Mike Harrison led to the question:
  - How to define RBM more generally than Harrison-Reiman. It is natural to consider d=2 first (d=1 is well known).
  - Challenges: non-smooth boundary, discontinuity in oblique reflection directions at corner, not symmetric in general

  - First natural question: Starting away from the origin, can Z ever reach the origin (with positive probability)?
    - S. R. S. Varadhan, while visiting Stanford on sabbatical in 1978, in a conversation with Marty Reiman and Mike Harrison, provided a sketch of an argument, including a key harmonic function.
    - Step 1. By Girsanov's theorem, we can assume without loss of generality that the underlying Brownian motion X has zero drift.
    - Step 2. Now make the standard change of variable that gives covariance matrix = I. That leads to a transformed problem in which standard Brownian motion is confined to a wedge by oblique reflection from each boundary ray, as shown on the next slide.

Key Harmonic Function

Step 3. Find a positive function f that solves the following PDE problem.



(1) $\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) f(r,\theta)$ for $r > 0$ and $0 \le \theta \le \xi$,

(2) $\left(\frac{1}{r}\frac{\partial}{\partial \theta} - \tan\gamma_1 \frac{\partial}{\partial r}\right) f(r,0) = 0$ for $r > 0$,

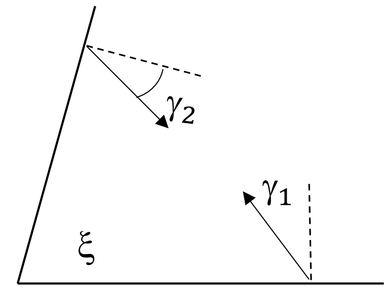
(3) $\left(\frac{1}{r}\frac{\partial}{\partial \theta} + \tan\gamma_2 \frac{\partial}{\partial r}\right) f(r,\xi) = 0$ for $r > 0$,

Step 4. Let $\alpha = \frac{\gamma_1+\gamma_2}{\xi}$. If $\alpha \neq 0$, then $f(r,\theta) = r^{\alpha}\cos(\alpha\theta - \gamma_1)$ is a solution.

If $\alpha = 0$, then $f(r,\theta) = \log r + \theta \tan\gamma_1$ is a solution.

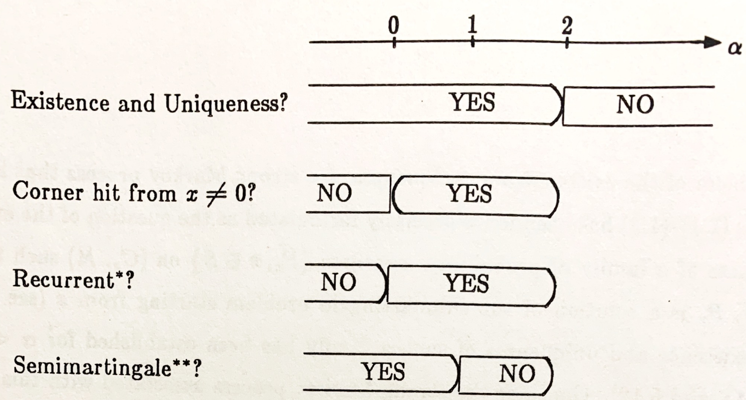
Step 5. It then follows from Ito's formula that the origin is hit with probability one $\Leftrightarrow \alpha > 0$.

Further Questions for RBM in a Wedge



- Can the RBM be extended uniquely beyond the first time of hitting the corner (or starting there)?
- When is the RBM a semimartingale?
- When is the RBM recurrent? What is an invariant measure?

$$\alpha = \frac{\gamma_1 + \gamma_2}{\xi}$$



* Density of invariant measure:

$$p(r,\theta) = \begin{cases} r^{-\alpha}\cos(\alpha\theta - \gamma_1) & \text{if } 0 < \alpha < 2 \\ 1 & \text{if } \alpha = 0 \end{cases}$$

**Unresolved case: $\alpha < 1$, $v_1 = -v_2$, $x = 0$. (Later resolved by D. De Blassie, 1990)

SRS Varadhan and RJ Williams, Brownian motion in a wedge with oblique reflection. CPAM (1985). (Submartingale problem)

RJ Williams, Recurrence classification and invariant measure for reflected Brownian motion in a wedge. Ann. Prob. (1985).

RJ Williams, Reflected Brownian motion in a wedge: Semimartingale property, ZW (1985).

Work on excursions, local times, stationary distributions, large deviations for d=2, by many other authors. See e.g., the references in "On the stationary distribution of reflected Brownian motion in a wedge: differential properties", M. Bousquet-Mélou, A. Elvey Price, S. Franceschi, C. Hardouin, K. Raschel, 2021. Generalizations, including to cusps (DeBlassie-Toby 1993, Burdzy-Toby 1995, Kurtz-Costantini 2018 ..).

Marty Reiman

Semimartingale Reflecting Brownian Motions in the Orthant
$Z = X + RL$ (existence and uniqueness)

Lisa Taylor

- Bell Labs preprint of Marty Reiman with some initial progress on the question of how much "pushing" is done by *L* at intersections of boundary faces.
- Led to Reiman, M.I., Williams, R.J. A boundary property of semimartingale reflecting Brownian motions. *PTRF (1988).*
  - There is weak existence of an SRBM for each starting point in the orthant only if *R* is completely-S.
  - Amount of pushing at boundary intersections is negligible:

$$\int_0^\infty \mathbf{1}_{\{Z_i(s)=0, Z_j(s)=0\}} dL_i(s) = 0 \quad \text{for any } j \neq i$$

- *R* being completely-S is not only necessary, but also sufficient for weak existence and uniqueness of an SRBM in the orthant, L.M Taylor and RJ Williams, PTRF (1993).
- Subsequently, Dai & Williams, Theor. Prob. Applic. (1995) generalized Taylor-W to a polyhedral state space.

- Work on solving the Skorokhod problem *z=x+Rℓ for x a continuous path, with connections to linear complementarity problems*, including Lions and Sznitman (1984), Mandelbaum & van der Heyden (unpublished), Bernard & El Kharroubi (1991), Dupuis and Ishii (1991).
- Subsequent generalizations in multiple directions including varying reflection field (Dupuis-Ishii, 1993), extended Skorokhod problem (Dupuis & Ramanan), Dirichlet processes, submartingale problem formulation and SDEs (Kang and Ramanan)…..

Mike Harrison

SRBMs: Stationary Distributions and Positive Recurrence

Paul Dupuis

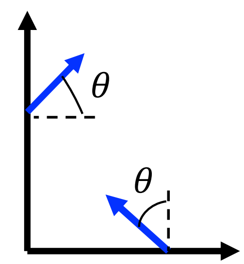
$$Lf = \frac{1}{2}\sum_{i,j=1}^{d} \Sigma_{ij}\frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_{i=1}^{d}\mu_i \frac{\partial f}{\partial x_i} \qquad D_i f = v_i \cdot \nabla f, \; i = 1, \ldots, d.$$

- Stationary Distributions for SRBM (BAR): Reiman-Harrison '81, Harrison-W '87, Harrison-Dai '92

$$\int_S Lf\, p_0\, dx + \sum_{i=1}^{d} \int_{F_i} D_i f\, p_i\, d\sigma_i = 0 \quad \text{for all} \quad f \in C_b^2(S).$$

Skew symmetry condition

$$2\Sigma = R \cdot \operatorname{diag}\Sigma + \operatorname{diag}\Sigma \cdot R^{\top}.$$



$\Sigma = I$

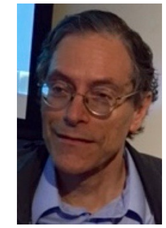
- Product form (exponential form) stationary distributions and skew symmetry condition
  - d=2: Harrison (1978), Harrison & Reiman (1981)
  - d$\geq$ **2**: Harrison & Williams (1987) x 2 and Williams (1987)
  - Connection of product form solution to quasireversibility, hybrid Atlas models and Internet congestion control

- Subsequent work by many on BAR: Dai, Kang-Ramanan, sum of exponentials: Dieker-Moriarty, skew symmetry: O'Connell & Ortmann, Franceschi & Raschel …

- Sufficient conditions for positive recurrence of SRBMs
  - Dupuis & Williams (1993): fluid model stability implies positive recurrence of SRBM. Further work by others, e.g., Atar, Budhiraja & Dupuis (2001), ..
  - Analogue for multiclass queueing networks: JG Dai, On positive Harris recurrence of multiclass queueing networks: A unified approach via fluid limit models (1995)

Multiclass Queueing Networks: Fluid and Diffusion Limits

Jim Dai



Maury Bramson

- Stability
  - Early to mid 1990s: surprising examples of instability of MQN and use of fluid models to prove stability
  - Counterexamples: Lu-Kumar (1991), Rybko-Stolyar (1992), Bramson (1994)
  - Stability via subcritical fluid models: Dai (1995), used by many authors

- Heavy traffic limits and diffusion approximations
  - Dai and Wang (1993): example of a multiclass queueing network with ill posed Brownian model
  - Bramson (1998) and Williams (1998): a modular approach to heavy traffic limits via asymptotic behavior of critical fluid models and state space collapse*

* MI **Reiman (1984): Some diffusion approximations with state space collapse.**

THANK YOU MERCI

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