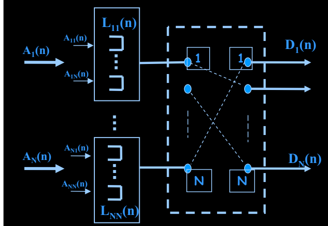
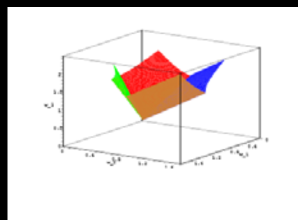
## Stochastic Processing Networks and SRBMs in Domains with Piecewise Smooth Boundaries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## OUTLINE

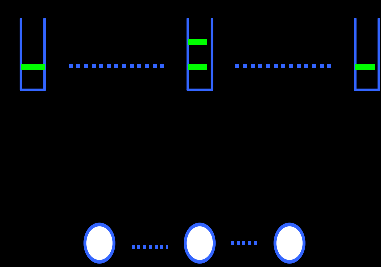
- Stochastic Processing Networks
- An Invariance Principle for SRBMs in Domains with Piecewise Smooth Boundary



# STOCHASTIC PROCESSING NETWORKS



## Stochastic Processing Networks (cf. Harrison '00)



$\mathbb{I}$ buffers (classes)

$\mathbb{J}$ activities

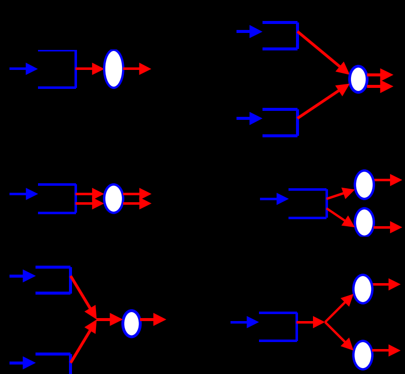
$\mathbb{K}$ servers (resources)

An activity consumes from certain classes, produces for certain (possibly different) classes, and uses certain servers.



## Stochastic Processing Networks

*Activities are Very General*



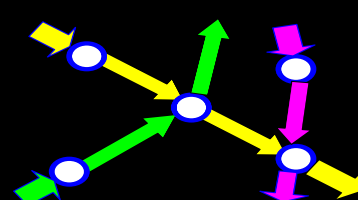
*Queueing network*

*Flexible servers, alternate routing*

*Simultaneous actions*



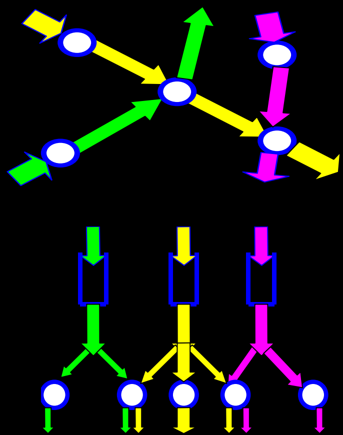
## Data Network (Roberts and Massoulie, '00)



Link

Route

## SPN with Simultaneous Resource Possession



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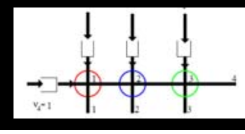
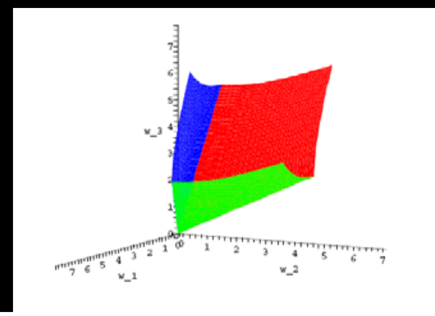
## Diffusion Workload Cone for a 3-node Linear Network under a Fair Bandwidth Sharing Policy



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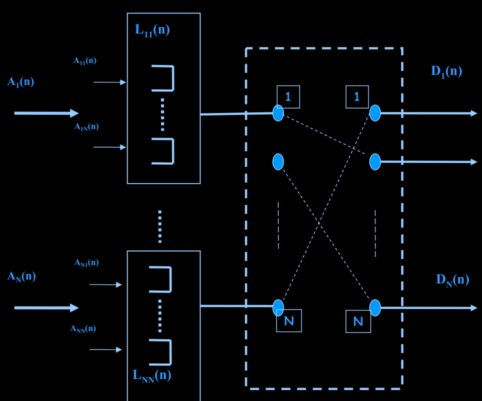
## NxN Input Queued Packet Switch: Prabhakar

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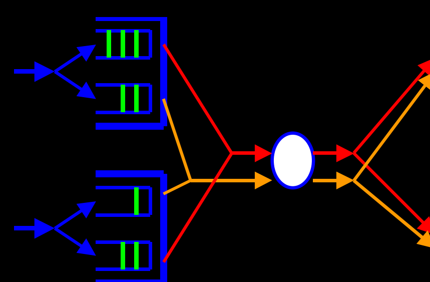
## 2x2 Input Queued Packet Switch



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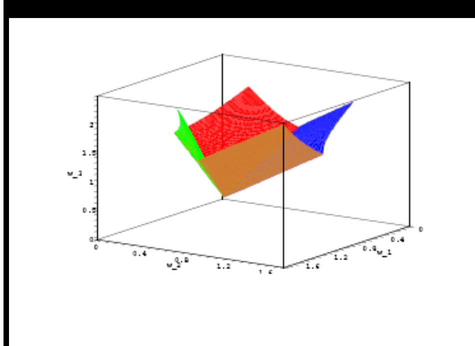
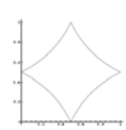
## Diffusion Workload Cone
(2 by 2 Switch using a Maximum Weight Matching algorithm)



Cross-section

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# SRBMs in Domains with Piecewise Smooth Boundaries

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# Outline





# SRBM Data

- $G$ a non-empty domain in $\mathbb{R}^d$ with piecewise smooth boundary:
  $G = \bigcap_{i\in\mathcal{I}} G_i$, where $\mathcal{I}$ is a finite index set and $G_i \neq \mathbb{R}^d$ is a domain with $C^1$ boundary, $i \in \mathcal{I}$.
- Denote the inward unit normal vector field on $\partial G_i$ by $n^i$, $i \in \mathcal{I}$
- $\gamma^i$ is a uniformly Lipschitz continuous unit length vector field on $\partial G_i,\ i \in \mathcal{I}$
- $\mu \in \mathbb{R}^d$, $\Gamma$ is a symmetric positive definite $d \times d$ matrix
- $\nu$ is a Borel probability measure on $\overline{G}$



# SRBM with data $(G, \mu, \Gamma, \{\gamma^i\}, \nu)$

An adapted, continuous $d$-dimensional process $W$ defined on some filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ such that

**(i)** $P$-a.s., for all $t \geq 0$, $W(t) \in \overline{G}$ and
$W(t) = W(0) + X(t) + \sum_{i\in\mathcal{I}} \int_{(0,t]} \gamma^i(W(s))dY_i(s)$,
and under $P$, $W(0)$ has distribution $\nu$,

**(ii)** under $P$, $X$ is a $(\mu, \Gamma)$-Brownian motion starting from the origin and $\{X(t) - \mu t, \mathcal{F}_t,\ t \geq 0\}$ is a martingale,

**(iii)** for each $i$, $Y_i$ is a continuous, increasing adapted, one-dimensional process starting from zero, such that $P$-a.s.,
$Y_i(t) = \int_{(0,t]} 1_{\{W(s)\in\partial G_i \cap \partial G\}} dY_i(s),\ t \geq 0.$



# Assumptions on $G$

**(A1)** For each $\varepsilon \in (0,1)$ there exists $R(\varepsilon) > 0$ such that for each $i \in \mathcal{I}$, $x \in \partial G_i \cap \partial G$ and $y \in \overline{G_i} \cap \overline{G}$ satisfying $||x - y|| < R(\varepsilon)$, we have

$$\langle n^i(x), y - x\rangle \geq -\varepsilon||y - x||.$$

**(A2)** $D(r) \to 0$ as $r \to 0$ where

$$D(r) = \sup_{\emptyset\neq\mathcal{J}\subset\mathcal{I}} \sup \Big\{ \text{dist}\Big(x, \bigcap_{j\in\mathcal{J}} (\partial G_j \cap \partial G)\Big) : \text{dist}(x, (\partial G_j \cap \partial G)) \leq r, \ \text{ for all } j \in \mathcal{J} \Big\}.$$



# Notes

1. $\mathcal{I}(x) = \{i \in \mathcal{I} : x \in \partial G_i\}$ is upper semi-continuous as a function of $x \in \partial G$.

2. If $G$ is bounded or convex, then (A1) holds.

3. If $G$ is bounded or a convex polyhedron, then (A2) holds.

# Assumptions on $\{\gamma^i\}$

**(A3)** There is $a > 0$ such that for each $x \in \partial G$, there are convex combinations $\gamma(x)$ of $\gamma^i(x)$ and $n(x)$ of $n^i(x)$ for $i \in \mathcal{I}(x)$ such that

(i) $\langle \gamma(x), n^i(x) \rangle > a$ for all $i \in \mathcal{I}(x)$,

(ii) $\langle n(x), \gamma^i(x) \rangle > a$ for all $i \in \mathcal{I}(x)$.



# Invariance Principle: Informally

Assume (A1)-(A3).

A sequence of processes that satisfies suitably perturbed versions of the SRBM conditions is $C$-tight.

In addition, if uniqueness in law holds for the SRBM, then the sequence of processes converges to the SRBM.

(Formal theorem —Kang-W '07)



# Applications: Existence

- Under (A1)-(A3), there exists an SRBM with the data $(G, \mu, \Gamma, \{\gamma^i\}, \nu)$.



# Applications: Approximation

A process that satisfies a perturbed version of the SRBM conditions is close in distribution to the SRBM under the following condition:

- $G$ is a convex polyhedron with minimal description. For each $i \in \mathcal{I}$, $\gamma^i$ is a constant vector field and $\{\gamma^i\}$ satisfies (A3).

  (Uniqueness in law of SRBMs holds by Dai-W '95)



# Approximation Continued ...

A process that satisfies a perturbed version of the SRBM conditions is close in distribution to the SRBM under the following condition:

- $G$ is a bounded domain with piecewise smooth boundary. The vector fields $\gamma^i, i \in \mathcal{I}$ are continuously differentiable with locally Lipschitz first partial derivatives and there is $a \in (0,1)$ such that for each $x \in \partial G$ there are non-negative $(b_i(x) : i \in \mathcal{I}(x))$ such that $\sum_{i \in \mathcal{I}(x)} b_i(x) = 1$ and for each $i \in \mathcal{I}(x)$:

$$b_i(x)\langle n^i(x), \gamma^i(x)\rangle \geq a + \sum_{j \in \mathcal{I}(x)\setminus\{i\}} b_j(x)|\langle n^j(x), \gamma^i(x)\rangle|.$$

  (Pathwise uniqueness of SRBM holds by Dupuis-Ishii '93)



# Invariance Principle: Hypotheses

Suppose that $\{\delta^n\}_{n=1}^\infty$ is a sequence of positive constants, and for each positive integer $n$, $d$-dimensional processes $W^n, X^n, \alpha^n$, and **I**-dimensional processes $Y^n, \beta^n$ are all defined on some probability space $(\Omega^n, \mathcal{F}^n, P^n)$ such that

**(i)** for $\widetilde{W}^n \equiv W^n + \alpha^n$, $P^n$-a.s.,
dist $\left(\widetilde{W}^n(t), \overline{G}\right) \leq \delta^n$ for all $t \geq 0$,

**(ii)** $P^n$-a.s.,
$W^n(t) = X^n(t) + \sum_{i \in \mathcal{I}} \int_{(0,t]} \gamma^{i,n}(W^n(s-), W^n(s))dY_i^n(s)$
for all $t \geq 0$, where $\gamma^{i,n} : \mathbb{R}^d \to \mathbb{R}^d$ is Borel measurable and $||\gamma^{i,n}(y,x)|| = 1$ for all $y, x \in \mathbb{R}^d$ and each $i \in \mathcal{I}$,

**(iii)** $X^n$ converges in distribution as $n \to \infty$ to a $(\mu, \Gamma)$-Brownian motion with initial distribution $\nu$,

# Hypotheses Continued ...

**(iv)** $\beta^n$ is locally of bounded variation and for $\widetilde{Y}^n \equiv Y^n + \beta^n$, $P^n$-a.s., for each $i \in \mathcal{I}$,

**(a)** $\widetilde{Y}_i^n(0) = 0,$

**(b)** $\widetilde{Y}_i^n$ is increasing and $\Delta \widetilde{Y}_i^n(t) \le \delta^n$ for all $t > 0,$

**(c)** $\widetilde{Y}_i^n(t) = \int_{(0,t]} 1_{\{\text{dist}\left(\widetilde{W}^n(s), \partial G_i \cap \partial G\right) \le \delta^n\}} d\widetilde{Y}_i^n(s)\ \forall\, t \ge 0,$

**(v)** $\delta^n \to 0$ as $n \to \infty$, and for each $\varepsilon > 0$, there is $\eta_\varepsilon > 0$ and $n_\varepsilon > 0$ such that for each $i \in \mathcal{I}$, $\|\gamma^{i,n}(y,x) - \gamma^i(x)\| < \varepsilon$ whenever $\|x - y\| < \eta_\varepsilon$ and $n \ge n_\varepsilon$,

**(vi)** $\alpha^n \to \mathbf{0}$ and $\mathcal{V}(\beta^n) \to \mathbf{0}$ in probability as $n \to \infty$, where for each $t \ge 0$, $\mathcal{V}(\beta^n)(t)$ is the total variation of $\beta^n$ on $[0,t]$.



# Invariance Principle (Kang-W '07)

Define $\mathcal{Z}^n = (W^n, X^n, Y^n)$ for each $n$. The sequence of processes $\{\mathcal{Z}^n\}_{n=1}^\infty$ is $C$-tight. Any (weak) limit point of this sequence is of the form $\mathcal{Z} = (W, X, Y)$ where all properties of the SRBM Definition hold, except possibly the martingale property, with $\mathcal{F}_t = \sigma\{\mathcal{Z}(s):\ 0 \le s \le t\},\ t \ge 0.$

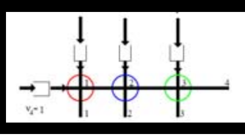
Furthermore, if the following conditions (vii) and (viii) hold, then $W^n \Rightarrow W$ as $n \to \infty$ where $W$ is an SRBM.

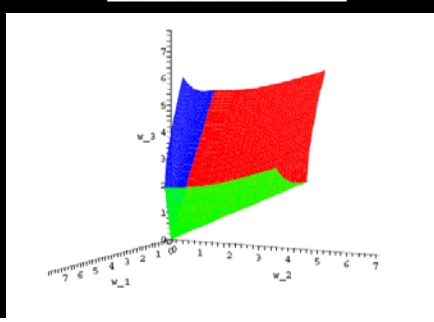
**(vii)** For each (weak) limit point $\mathcal{Z} = (W, X, Y)$ of $\{\mathcal{Z}^n\}_{n=1}^\infty$, $\{X(t) - \mu t,\ \mathcal{F}_t,\ t \ge 0\}$ is a martingale.

**(viii)** If a process $W$ satisfies the SRBM Definition, then the law of $W$ is unique.



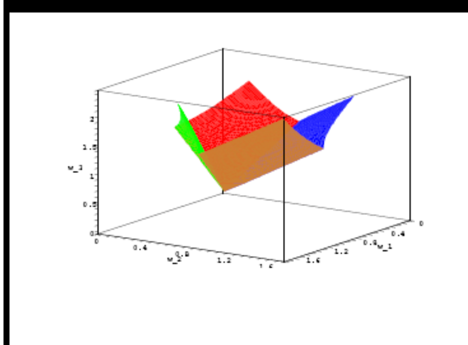
## Diffusion Workload Cone for a 3-node Linear Network under a Fair Bandwidth Sharing Policy

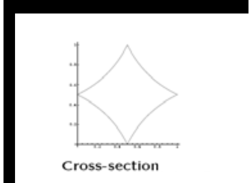







## Diffusion Workload Cone (2 by 2 Switch using a Maximum Weight Matching algorithm)





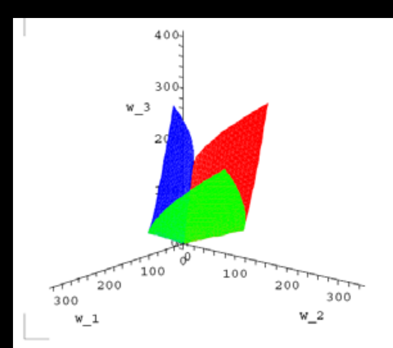



## Open Problems

- More general sufficient conditions for (weak) uniqueness of SRBMs
- Treatment of domains with cusp-like boundary interfaces
- Treatment of domains with smooth meetings of boundaries

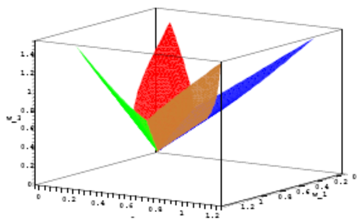


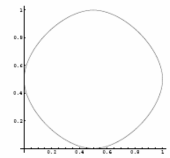
## Diffusion Workload Cone for a 3-node Linear Network under another Fair Bandwidth Sharing Policy



# Diffusion Workload Cone

(2 by 2 Switch using another Maximum Weight Matching algorithm)





Cross-section