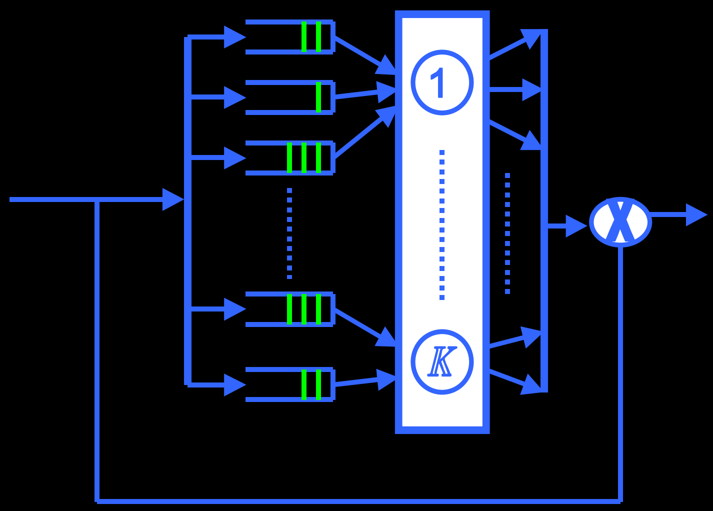
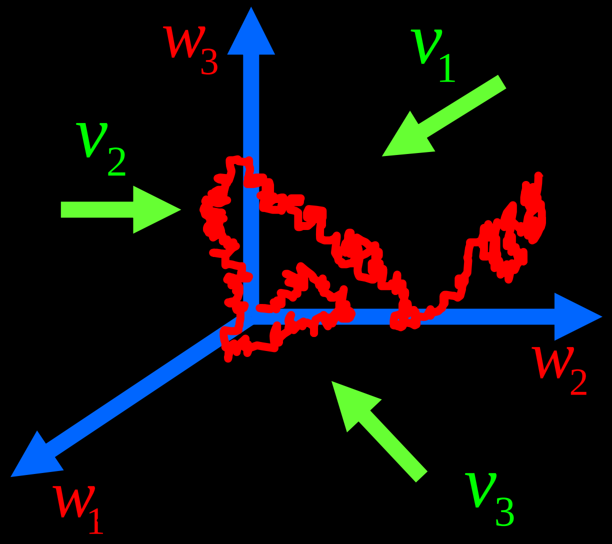
MULTICLASS QUEUEING NETWORKS
AND
SRBMS IN THE ORTHANT





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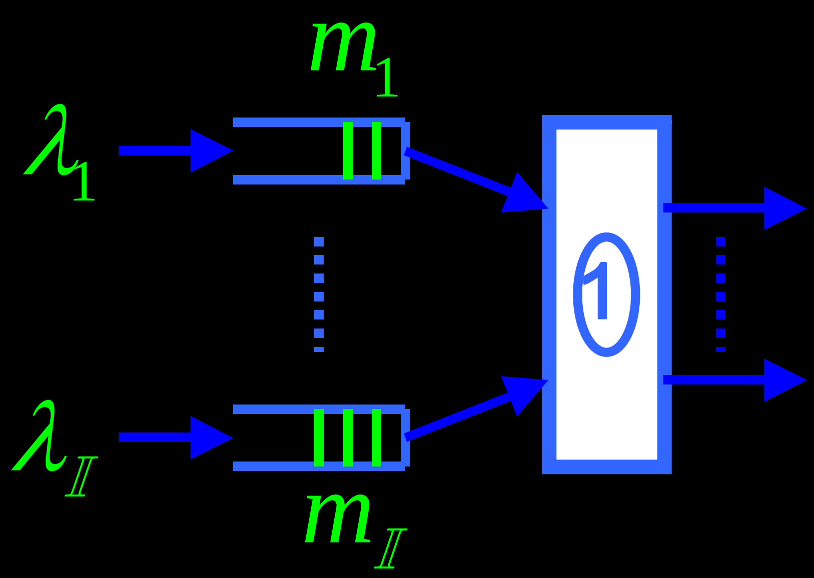
Outline

- SIMPLE MULTICLASS EXAMPLE
- OPEN MULTICLASS HL NETWORK (CONJECTURES)
- HISTORY
- SEMIMARTINGALE REFLECTING BROWNIAN MOTIONS IN THE ORTHANT
- FURTHER DEVELOPMENTS

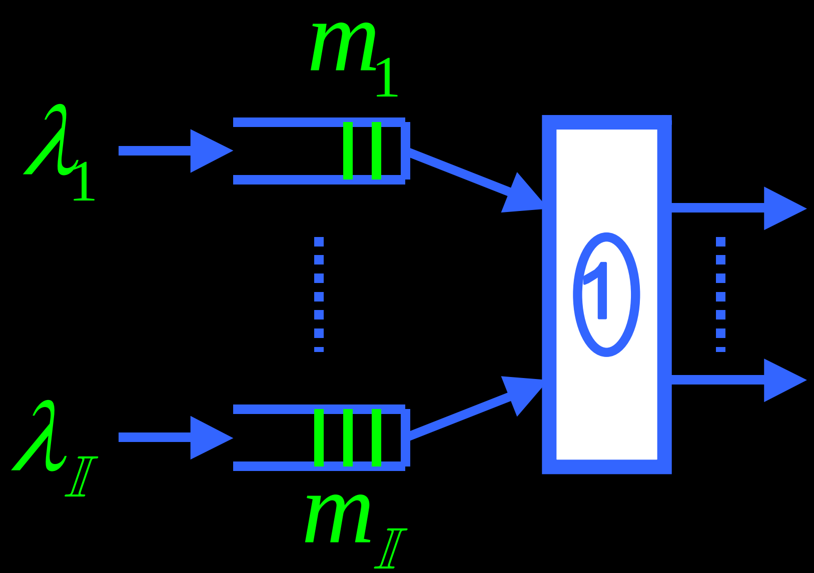
SIMPLE MULTICLASS EXAMPLE

Multiclass FIFO Station



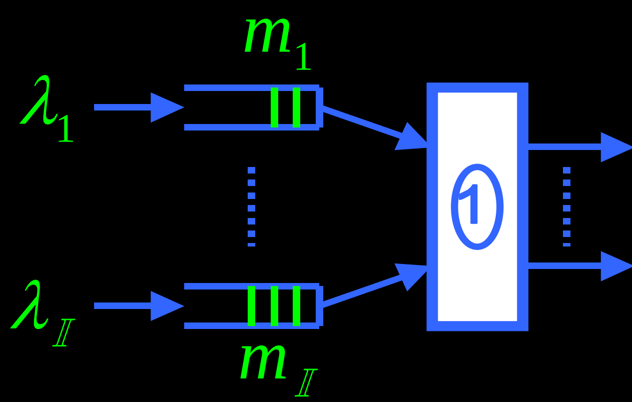
- Renewal arrivals to class *i* at rate λ_i
- i.i.d. service times for class *i*, mean m_i
- Service discipline: FIFO across all classes

Performance Processes



- Queuelength for class *i*: Q_i
- Workload: W
- Idletime: Y

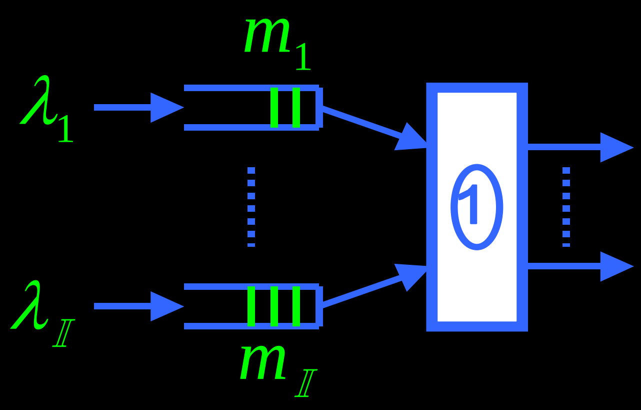
Stability



- Traffic Intensity $\rho_1 = \sum_{i=1}^{I} \lambda_i m_i$

- Stability iff $\rho_1 < 1$

Stability

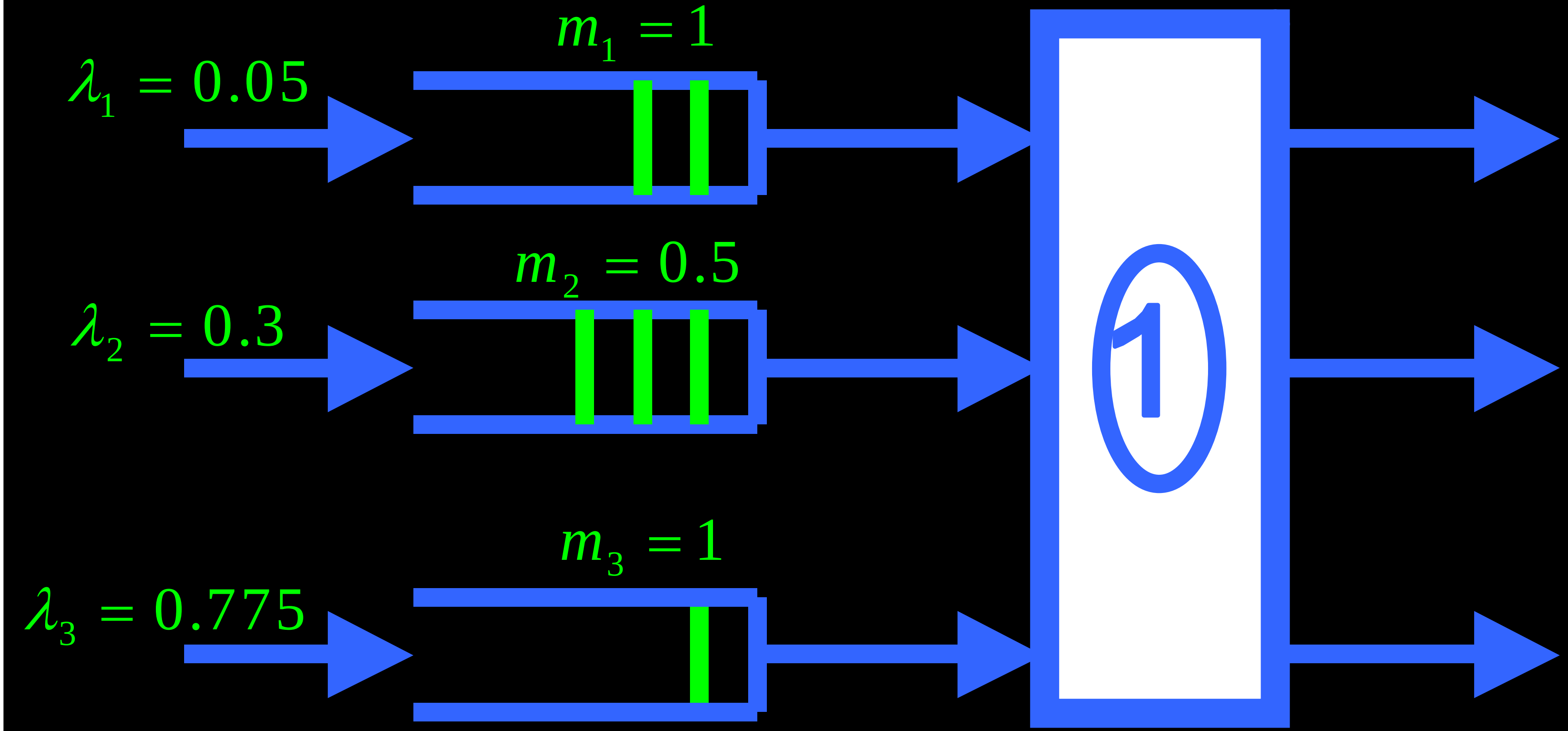


- Traffic Intensity $\rho_1 = \sum_{i=1}^{I} \lambda_i m_i$

- Stability iff $\rho_1 < 1$
- Heavy traffic $\rho_1 \approx 1$

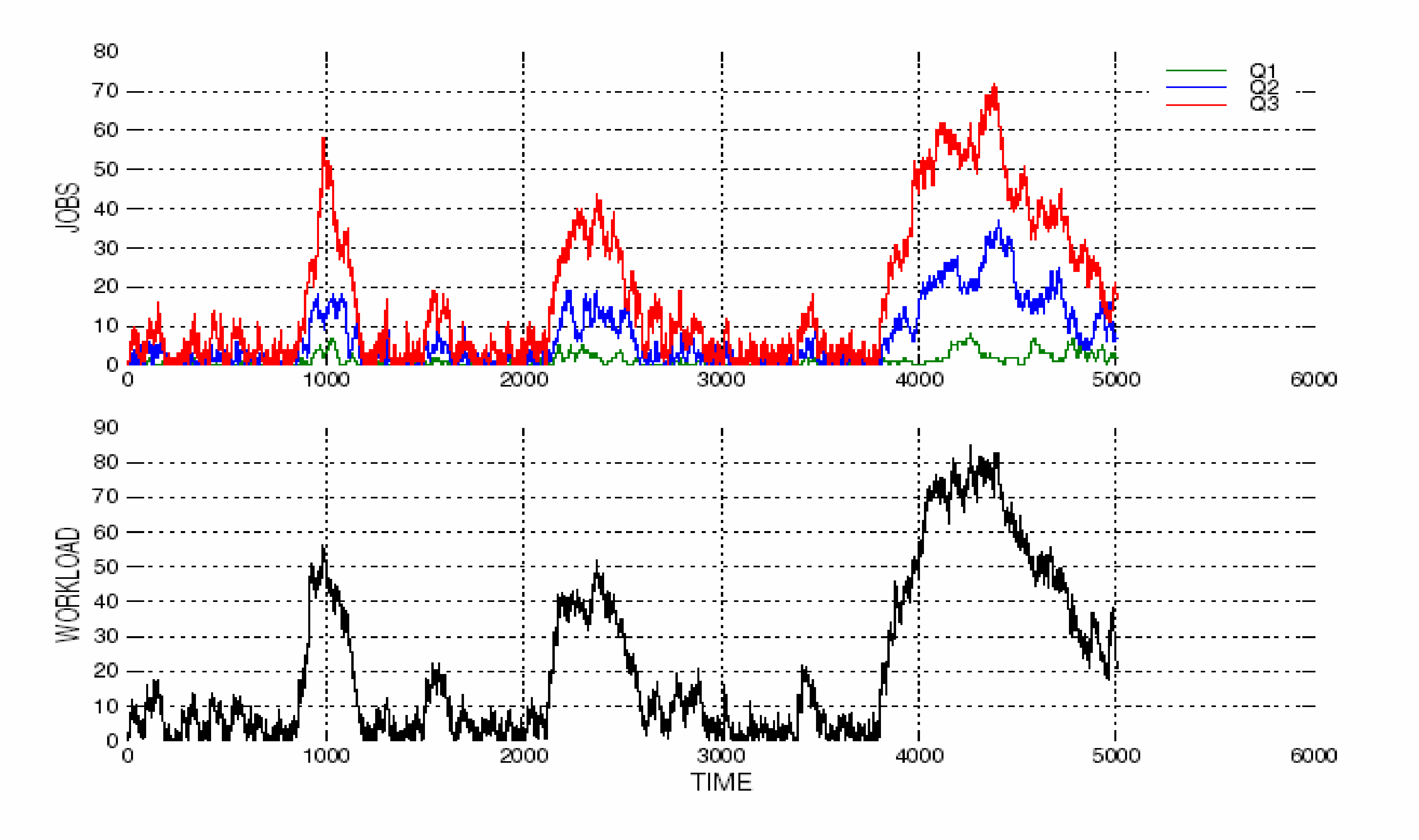
Simulation of a Multiclass FIFO queue

(Poisson arrivals, exponential service times)

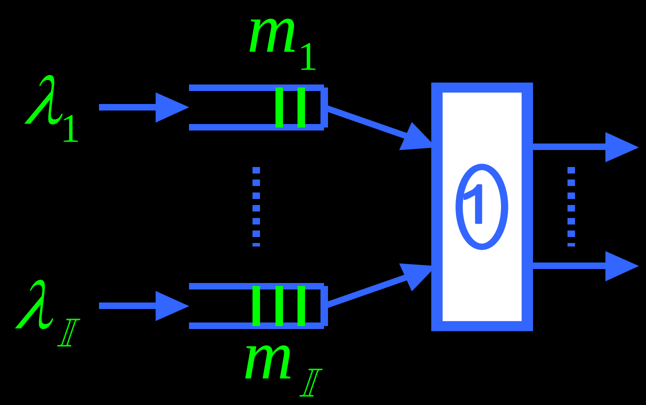


$$\rho_1 = 0.975$$

Simulation of a Multiclass FIFO queue



Stability



- Traffic Intensity $\rho_1 = \sum_{i=1}^{\mathbb{I}} \lambda_i m_i$

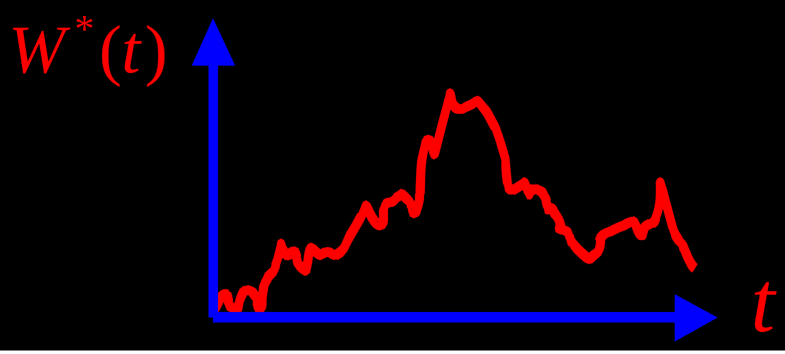
- Stability iff $\rho_1 < 1$
- Heavy traffic $\rho_1 \approx 1$ (assume $\rho_1 = 1$ for simplicity)

Heavy Traffic Diffusion Approximation

$$\hat{W}^r(t) = W(r^2 t)/r, \quad \hat{Y}^r(t) = Y(r^2 t)/r,$$

$$\hat{Q}_i^r(t) = Q_i(r^2 t)/r, \; i = 1, \ldots, \mathbb{I}$$

<u>Theorem (Whitt '71)</u> $(\hat{W}^r, \hat{Y}^r, \hat{Q}^r) \Rightarrow (W^*, Y^*, Q^*)$ where W^* is a one-dimensional reflecting Brownian motion with local time Y^* and $Q^* = \lambda W^*$ (state space collapse).



$$W^*(t) = X^*(t) + Y^*(t)$$

$$Y^*(t) = \sup\{-X^*(s) : 0 \le s \le t\}$$

$$X^* = \text{Brownian motion}$$

OPEN MULTICLASS

HL NETWORK

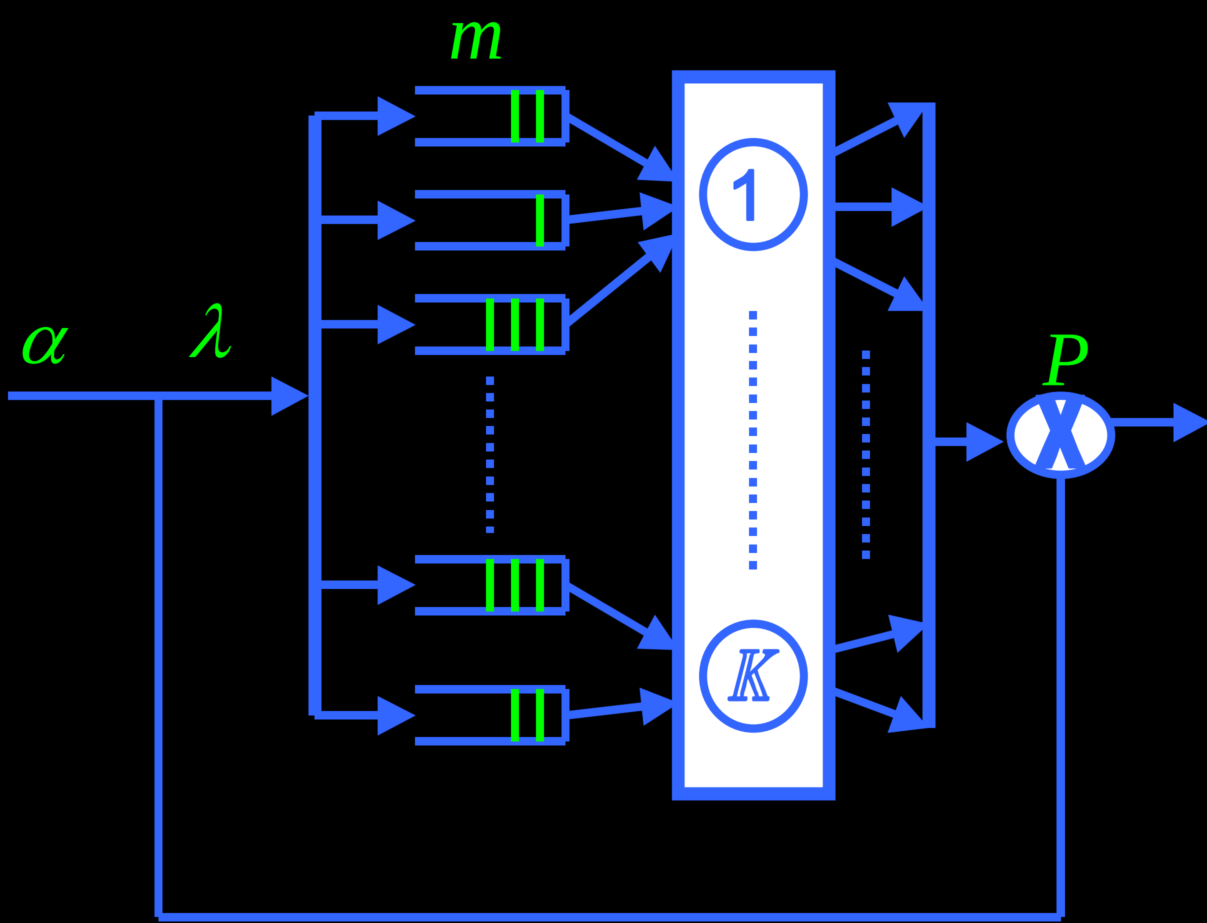
(CONJECTURES)

Assumptions

- Open: jobs enter the system from outside and eventually leave the network. Assume infinite capacity buffers.
- HL: jobs within a buffer are stored in the order in which they arrived and service is always given to the job at the head-of-the-line. Also, the discipline is non-idling.
- Primitive arrival, service and routing processes are assumed to satisfy functional central limit theorems.

Open Multiclass HL Queueing Network

First order parameters



$$\lambda = \alpha + P'\lambda \qquad \rho_k = \sum_{i \in k} \lambda_i m_i, \; k = 1, \ldots, \mathbb{K}$$

Natural Conjectures

- Stability: Network is stable provided

$$\rho_k < 1 \quad \text{for each} \quad k = 1, \ldots, \mathbb{K}$$

- Heavy traffic diffusion approximation:

If $\rho_k \approx 1,\ k = 1, \ldots, \mathbb{K},$ then $(\hat{W}^r, \hat{Y}^r, \hat{Q}^r) \approx (W^*, Y^*, Q^*)$ where $Q^* = \Delta W^*$ for some $\mathbb{I}$x$\mathbb{K}$ lifting matrix Δ

(that depends on the HL service discipline), and

$W^* = X^* + RY^*$ is a reflecting Brownian motion (RBM) in the $\mathbb{K}$-dimensional orthant.

HISTORY

Affirmative Answers

(Refs. are for diffusion approximations through early 1990s)

- SINGLE CLASS (FIFO):
  – Single station: Borovkov ('67), Iglehart-Whitt ('70)
  – Acyclic network: Iglehart-Whitt ('70), Tandem queue: Harrison ('78)
  – Network: Reiman ('84), Chen-Mandelbaum ('91)
- MULTICLASS:
  – Single station, priorities: Whitt ('71), Harrison ('73)
  – Network, priorities: Johnson ('83, SP), Peterson ('91, feedforward)
  – Single station, feedback, round robin & FIFO: Reiman ('88), Dai-Kurtz ('95)

Rely on continuous mapping construction of SRBM and do not cover multiclass networks with general feedback.

Counterexamples

(two-stations, deterministic routing)

- STABILITY
  - Kumar and Seidman ('90): clearing policy.
  - Lu and Kumar ('91): static priorities, deterministic interarrival and service times.
  - Rybko and Stolyar ('92): static priorities, exponential interarrival and service times. (See also Botvitch and Zamyatin ('92))
  - Seidman ('94): FIFO, deterministic interarrival and service times.
  - Bramson ('94): FIFO, exponential interarrival and service times.
- DIFFUSION APPROXIMATION
  - Dai-Wang ('93): FIFO, exponential interarrival and service times.

HL MQN: Sufficient Conditions

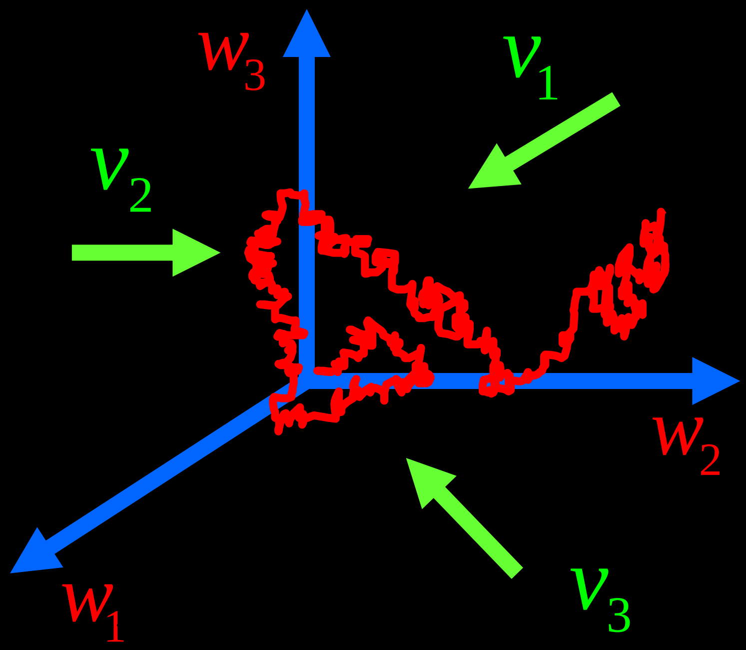
- STABILITY
  - Subcritical fluid models

- PERFORMANCE ANALYSIS (in heavy traffic)
  - Reflecting diffusions and state space collapse via critical fluid models

SEMIMARTINGALE REFLECTING BROWNIAN MOTIONS (SRBMs)

SRBM DATA

- State space: $\mathbb{R}_{+}^{\mathbb{K}}$
- Brownian statistics: drift θ, covariance matrix Γ
- Reflection matrix: $R = \left(v_1, \ldots, v_{\mathbb{K}}\right)$



SRBM DEFINITION (w/starting point x_0)

A continuous $\mathbb{K}$-dimensional process W such that

(i) $W = X + RY$

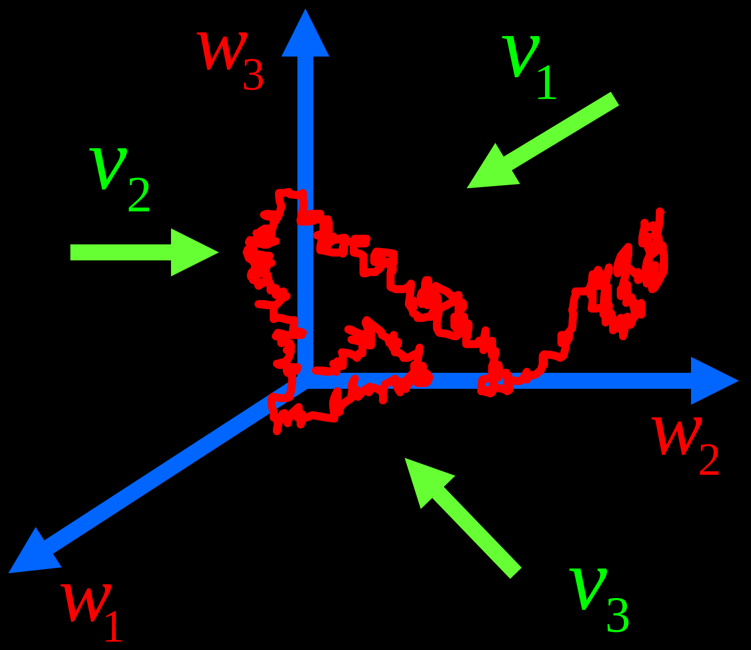
(ii) W has paths in $\mathbb{R}_+^{\mathbb{K}}$

(iii) for *k*=1,...,$\mathbb{K}$, $Y_k(0) = 0$, Y_k is continuous, non-decreasing, and it can increase only when $W_k = 0$

(iv) X is a (θ, Γ) BM s.t. $X(0) = x_0$, $\{X(t) - \theta t, t \geq 0\}$ is a martingale relative to the filtration generated by (W, X, Y)

Necessary Condition for Existence

Defn: R is completely-S iff for each principal submatrix $\tilde{R}$ of R there is $\tilde{y} > 0 : \tilde{R}\tilde{y} > 0$



$$R = \left(v_1, \ldots, v_K\right)$$

Existence and Uniqueness in Law

Theorem (Reiman-W '88, Taylor-W '93)
There is an SRBM W starting from each point x_0 in $\mathbb{R}_+^{\mathbb{K}}$ iff R is completely-*S*. In this case, each such SRBM is unique in law and these laws define a continuous strong Markov process.

Oscillation Inequality

Assume that *R* is completely-*S*. There is a constant *C>0* such that whenever $\delta > 0,\ \ 0 \le t_1 < t_2 < \infty,$ and w, x, y are r.c.l.l. satisfying

(i) $w(t) = x(t) + Ry(t)$ for $t \in [t_1, t_2]$

(ii) w lives in $\mathbb{R}_+^{\mathbb{K}}$

(iii) for *k*=1,...,$\mathbb{K}$, $y_k(t_1) \ge 0,\ y_k$ is continuous, non-decreasing, and can increase only when $w_k < \delta$,

Then

$$Osc(w,[t_1,t_2]) + Osc(y,[t_1,t_2]) \le C(Osc(x,[t_1,t_2]) + \delta)$$

Cts case: Bernard-El Kharroubi '91, discts case: W '98

Analysis of multidimensional SRBMs

- Sufficient conditions for positive recurrence

  Dupuis-W ‘94, Chen ‘96, Budhiraja-Dupuis ‘99, El Kharroubi-Ben Tahar-Yaacoubi ‘00
- Stationary distribution
  - *Characterization:* Harrison-W ‘87, Dai-Harrison ‘92, Dai-Kurtz ‘98
  - *Analytic solutions -two-dimensions:* Foddy ‘84, Trefethen-W ’86, Harrison ‘06

    -product form: Harrison-W ‘87
  - *Numerical methods:* Dai-Harrison ‘91,’92, Shen-Chen-Dai-Dai ‘02, Schwerer ‘01
- Large deviations

  Majewski ‘98,’00, Avram-Dai-Hasenbein ’01, Dupuis-Ramanan ’02,

Some Related Work on RBMs & Queueing Networks

- Capacitated queues (convex polyhedral domains)
  - Dai-Williams ’95, Dai-Dai ‘99

- HT limits that are not SRBMs (& have no state space collapse)
  - Single station-polling: Coffman-Puhalskii-Reiman ‘95
  - Dynamic HLPS: Dupuis-Ramanan ’99, Ramanan-Reiman ‘03

- Non-HL service disciplines
  (Markovian state descriptor is typically infinite dimensional)
  - LIFO preemptive resume: Single station: Limic ‘00, ‘01
  - Processor sharing: Single station (Gromoll-Puha-W ‘01, Puha-W ‘03, Gromoll ‘03); network (stability: Bramson ‘04)
  - EDF: Single station (Doytchinov-Lehoczky-Shreve ‘01), acyclic network (Kruk-Lehoczky-Shreve-Yeung ‘03), network (stability: Bramson ‘01)

PERSPECTIVE

	MQN	SPN
HL	Sufficient conditions for stability and diffusion approximations	e.g., parallel server system, packet switch
Non-HL	e.g., LIFO, Processor Sharing (single station, PS: network stability)	e.g., Internet congestion control / bandwidth sharing model