Probability and
Stochastics Series

Topics in Contemporary Probability and Its Applications

Edited by
J. Laurie Snell

CRC Press
Boca Raton New York London Tokyo

Library of Congress Cataloging-in-Publication Data

Topics in contemporary probability and its application / [compiled
by] J. Laurie Snell
p. cm. -- (Probability and stochastics series)
Includes bibliographical references and index.
ISBN 0-8493-8073-1 (alk. paper).
1. Stochastic processes. 2. Probabilities. I. Snell, J. Laurie
(James Laurie), 1925— . II. Series.
QA274.T65 1994
519.2--dc20

94-23472
CIP

Direct all inquiries to CRC Press, Inc., 2000 Corporate Blvd., N.W., Boca Raton, Florida 33431.

International Standard Book Number 0-8493-8073-1
Library of Congress Card Number 94-23472
1 2 3 4 5 6 7 8 9 0
Printed on acid-free paper

Chapter 3

SOME CONNECTIONS BETWEEN BROWNIAN MOTION AND ANALYSIS VIA STOCHASTIC CALCULUS*

R. J. Williams[†]
Department of Mathematics
University of California, San Diego

ABSTRACT

In this paper, some connections between the fundamental stochastic process Brownian motion and the mathematical subject of analysis are made using stochastic calculus. This calculus which was introduced by K. Itô enables one to compute with functions of Brownian motion. A distinctive feature of stochastic calculus is that the change of variables formula is different from that in ordinary Newton calculus because the sample paths of Brownian motion are of unbounded variation. In this note, some basic aspects of stochastic calculus are explained first. Then this calculus is used as a tool to make connections between Brownian motion and the following problems in analysis: the classical Dirichlet problem, the Schrödinger equation, and Laplace's equation with oblique derivative boundary conditions in a quadrant. The latter is relevant to the study of approximations to two station queueing systems in heavy traffic. References to some of the multitude of other properties and applications of Brownian motion are included at the end of this paper.

1. INTRODUCTION

Brownian motion in $\mathbb{R}^d$ $(d \geq 1)$ is a fundamental stochastic process because it lies at the intersection of many different topics in probability, analysis, and applied stochastics. In particular, if $B = \{B(t),\ t \geq 0\}$ is a Brownian motion in $\mathbb{R}^d$ that starts from the origin, then (i) B is a limit of renormalized simple symmetric random walks, (ii) B has the self-similarity property that it is equal in distribution to $\{\lambda^{-\frac{1}{2}}B(\lambda t),\ t \geq 0\}$ for any $\lambda > 0$, (iii) B has

*Part of the material discussed in this article was presented in an MAA Invited Lecture at the Annual Joint Mathematics Meeting held in Phoenix, Arizona, in January 1989.

[†]This article was written while the author was visiting the Technion, Israel, and was supported in part by an Alfred P. Sloan Fellowship.

0-8493-8073-1/95/$0.00+$.50

independent components, stationary independent increments, and continuous sample paths, (iv) B is a Gaussian process, (v) B is a martingale with respect to its own filtration, (vi) any bounded harmonic function of B yields a martingale, (vii) B is a time-homogeneous Markov process with transition probability densities that satisfy the heat equation, and (viii) B plays the role of the key source of randomness in many models arising in applications in the physical, biological and social sciences.

This note focuses on some connections between Brownian motion and analysis, for which property (vi) above is a prototype. Some of the examples presented here are motivated by problems arising in applications. A key tool in the discussion that follows is a *stochastic calculus* that enables one to compute with functions of Brownian motion. A feature of this calculus is that the change of variables formula is different from the one in ordinary Newton calculus, because the sample paths of Brownian motion are of unbounded variation. Before the rudimentary aspects of stochastic calculus are described, the reader is reminded of some aspects of ordinary Riemann-Stieltjes integration. There is no need to consider Lebesgue-Stieltjes integrals because all of the processes considered here have continuous sample paths.

Let $f, g : \mathbb{R}_+ \to \mathbb{R}$ be continuous functions, and suppose that g is also locally of bounded variation, i.e., g is of bounded variation on each compact interval in $\mathbb{R}_+$. Under the rules of ordinary Riemann-Stieltjes integration, one can define

$$\int_0^t f(s)dg(s) = \lim_{n\to\infty} \sum_{t_i^n, t_{i+1}^n \in \pi_n} f(\tilde{t}_i^n)(g(t_{i+1}^n) - g(t_i^n)), \tag{1}$$

where for each n, $\pi_n \equiv \{t_0^n, t_1^n, \ldots, t_n^n\}$ is a partition of $[0,t]$ such that $0 = t_0^n < t_1^n < \cdots < t_n^n = t$, $\tilde{t}_i^n \in [t_i^n, t_{i+1}^n]$ for each $i \in \{0, 1, \ldots, n-1\}$, and $|\pi_n| \equiv \max_{i=0}^{n-1} |t_{i+1}^n - t_i^n| \to 0$ as $n \to \infty$. Furthermore, by requiring that the following integration by parts formula hold,

$$f(t)g(t) - f(0)g(0) = \int_0^t f(s)dg(s) + \int_0^t g(s)df(s), \tag{2}$$

one can define $\int_0^t g(s)df(s)$, because all of the other entities in (2) are well defined. Note that f need not be of bounded variation. However, if neither f nor g is locally of bounded variation, then in general one cannot make sense of the deterministic integral $\int_0^t f(s)dg(s)$. Finally, if g is locally of bounded variation and $F : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function, then one has the change of variables formula

$$F(g(t)) = F(g(0)) + \int_0^t F'(g(s))dg(s). \tag{3}$$

Now, consider a Brownian motion B in $\mathbb{R}$. Since the sample paths of B are continuous, for a given sample path of B, one can take $f(s) = B(s)$ for

$s \geq 0$ in the above, i.e., for a given ω one can consider $f(s) = B(s, \omega)$. Then one can use the above to define $\int_0^t g(s)dB(s)$ for any continuous function g that is locally of bounded variation. Indeed, g can even be a sample path of a stochastic process having these properties sample path by sample path. However, one cannot use this procedure to define even such simple integrals as $\int_0^t B(s)dB(s)$, because the sample paths of Brownian motion are *not* locally of bounded variation. Indeed, they are only locally of finite *quadratic variation*, i.e., for each $t \in \mathbb{R}_+$ and sequence $\{\pi_n, n = 1, 2, \ldots\}$ of partitions of $[0, t]$ as described before,

$$[B]_t \equiv \lim_{n\to\infty} \sum_{t_i^n, t_{i+1}^n \in \pi_n} (B(t_{i+1}^n) - B(t_i^n))^2 \tag{4}$$

exists as a non-trivial limit in probability. Indeed, by a bare-hands calculation one can show that $[B]_t = t$. For many practical purposes one would like to be able to define integrals of the form $\int_0^t f(B(s))dB(s)$ for continuous functions $f : \mathbb{R} \to \mathbb{R}$. In particular, such integrals play an essential role in the development of a change of variables formula for sufficiently differentiable functions of B. With regard to this, note that by use of a telescoping series,

$$\begin{aligned}(B(t))^2 = (B(0))^2 \quad &+ \sum_{t_i^n, t_{i+1}^n \in \pi_n} 2B(t_i^n)(B(t_{i+1}^n) - B(t_i^n)) \\ &+ \sum_{t_i^n, t_{i+1}^n \in \pi_n} (B(t_{i+1}^n) - B(t_i^n))^2 \end{aligned} \tag{5}$$

where by (4) the last sum on the right tends to t in probability as $n \to \infty$. Thus, if one defines $\int_0^t B(s)dB(s)$ to equal the limit in probability of $\sum_{t_i^n, t_{i+1}^n \in \pi_n} B(t_i^n)(B(t_{i+1}^n) - B(t_i^n))$ as $n \to \infty$, then one obtains from (5) that

$$(B(t))^2 = (B(0))^2 + 2\int_0^t B(s)dB(s) + t. \tag{6}$$

This suggests that a change of variables formula for B does not have the same form as (3) in general, due to an extra contribution from the quadratic variation of B. In fact, one can make sense of integrals such as $\int_0^t B(s)dB(s)$ as limits in probability of approximating sums of the form indicated above, and there is a change of variables formula for *twice* continuously differentiable functions of B. Since the examples that follow involve d-dimensional Brownian motions with various starting points and sometimes additional processes with sample paths that are locally of bounded variation, stochastic integrals and the associated change of variables formula are described below in sufficient generality to accommodate these examples. First, a definition of d-dimensional Brownian motion (or equivalently, Brownian motion in $\mathbb{R}^d$) is given.

A d-dimensional Brownian motion that starts from the origin is a stochastic process $B \equiv \{B(t) : t \geq 0\}$ with continuous sample paths and independent components $B_1, \ldots, B_d$, such that for each $j \in \{1, \ldots, d\}$, B_j is a one-dimensional Brownian motion characterized by (i)–(iii) below:

(*i*) for any $0 = t_0 < t_1 < \ldots < t_\ell < \infty$, $\{B_j(t_k) - B_j(t_{k-1}),\ k = 1, \ldots, \ell\}$ are independent random variables,
(*ii*) for any $0 \leq s < t < \infty$, $B_j(t) - B_j(s)$ is a normally distributed random variable with mean zero and variance $t - s$,
(*iii*) $B_j(0) = 0$.

A d-dimensional Brownian motion that starts from $x \in \mathbb{R}^d$ is obtained by replacing (iii) by (iii′): $B_j(0) = x_j$, the j^{th} component of x. In the sequel, B denotes a d-dimensional Brownian motion starting from some $x \in \mathbb{R}^d$, defined on a complete probability space $(\Omega, \mathcal{F}, P_x)$. Expectations with respect to P_x are denoted by E_x. For each $t \geq 0$, $\mathcal{F}_t \equiv \sigma\{B(s) : 0 \leq s \leq t\}^{\sim}$, the σ-field generated by B up to time t and augmented (denoted by the tilde) by the P_x-null sets in $\mathcal{F}$. Then, $\{B(t), \mathcal{F}_t, t \geq 0\}$ is a martingale [see (i)–(iii) below for a definition]. Let m be a non-negative integer and let $Y = \{Y(t), t \geq 0\}$ be an m-dimensional stochastic process defined on $(\Omega, \mathcal{F}, P_x)$, such that Y has continuous sample paths and for each t, $Y(t)$ is measurable with respect to $\mathcal{F}_t$, i.e., $Y(t)$ is measurable as a function from $(\Omega, \mathcal{F}_t)$ into $(\mathbb{R}^m, \mathcal{B}^m)$ where $\mathcal{B}^m$ denotes the family of Borel sets in $\mathbb{R}^m$. When referring to the latter property, one says that Y is *adapted* to $\{\mathcal{F}_t : t \geq 0\}$. It is further assumed that the sample paths of Y are locally of bounded variation. Let $f : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d$ be a continuous function. Then for each $t \in \mathbb{R}_+$,

$$
\begin{aligned}
M(t) &\equiv \int_0^t f(B(s), Y(s)) \cdot dB(s) \\
&= \lim_{n\to\infty} \sum_{j=1}^{d} \sum_{t_i^n, t_{i+1}^n \in \pi_n} f_j(B(t_i^n), Y(t_i^n))(B_j(t_{i+1}^n) - B_j(t_i^n)) \qquad (7)
\end{aligned}
$$

can be shown to exist as a limit in probability, where $\{\pi_n, n = 1, 2, \ldots\}$ is a sequence of partitions of $[0, t]$ as described before. Moreover, M can be taken to have continuous sample paths and if f is bounded, then $\{M(t), \mathcal{F}_t, t \geq 0\}$ is a martingale, i.e.,

(*i*) $M(t)$ is $\mathcal{F}_t$-measurable for each $t \geq 0$,
(*ii*) $E_x[|M(t)|] < \infty$ for each $t \geq 0$,
(*iii*) $E_x[M(t) \,|\, \mathcal{F}_s] = M(s)$ for all $0 \leq s < t < \infty$,

where $E_x[\ \cdot \ |\mathcal{F}_s]$ denotes conditional expectation given $\mathcal{F}_s$. In particular, $E_x[M(t)] = E_x[M(0)] = 0$ for all $t \geq 0$. In order to obtain the martingale property for M (which is inherited from that of B), it is important that one use $f_j(B, Y)$ evaluated at the *left* end-point of the intervals $[t_i^n, t_{i+1}^n]$ in the approximating sums of (7). Using other points in the interval can yield a different (non-martingale) value for the integral. This is in marked contrast to the situation for Riemann-Stieltjes integrals. Now, if $F : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}$ is such that $F = F(b, y)$ is twice continuously differentiable in $b \in \mathbb{R}^d$ and once continuously differentiable in $y \in \mathbb{R}^m$, then it can be shown that P_x-a.s. for all $t \geq 0$,

$$\begin{aligned} F(B(t), Y(t)) &= F(B(0), Y(0)) + \textstyle\int_0^t \nabla_b F(B(s), Y(s)) \cdot dB(s) \\ &+ \textstyle\int_0^t \nabla_y F(B(s), Y(s)) \cdot dY(s) + \frac{1}{2} \int_0^t \Delta_b F(B(s), Y(s)) ds, \end{aligned} \tag{8}$$

where $\nabla_b F$ denotes the gradient of F with respect to its first d arguments, $\nabla_y F$ denotes the gradient of F with respect to its last m arguments, $\Delta_b F$ denotes the d-dimensional Laplacian of F with respect to its first d arguments, and the first integral in (8) is defined as a limit in probability as per (7), and the second integral is a sum of m Riemann-Stieltjes integrals defined path-by-path with respect to the locally bounded variation sample paths of $Y_1, \ldots, Y_m$, i.e.,

$$\int_0^t \nabla_y F(B(s), Y(s)) \cdot dY(s) = \sum_{k=1}^m \int_0^t \frac{\partial F}{\partial y_k}(B(s), Y(s)) dY_k(s).$$

The last integral in (8) is defined path-by-path as an ordinary Riemann integral. Formula (8) is a version of Itô's change of variables formula in stochastic calculus. Note that this differs from the formula in ordinary Newton calculus by the addition of the last term in (8) that arises because the paths of Brownian motion are locally of finite quadratic variation rather than being locally of bounded variation. If $d = 1$ and $m = 0$ then (8) simplifies to

$$F(B(t)) = F(B(0)) + \int_0^t F'(B(s)) dB(s) + \frac{1}{2} \int_0^t F''(B(s)) ds, \tag{9}$$

and in particular, if $F(b) = b^2$ then one recovers (6). For a justification of (7) and (8) the reader is referred to [4]. Here the use of (8) will be illustrated with some examples. The first of these is a classical application to the Dirichlet problem.

2. DIRICHLET PROBLEM

Let D be a bounded domain in $\mathbb{R}^d$ with boundary ∂D. Let f be a continuous real-valued function defined on ∂D. Consider solutions $u \in C^2(D) \cap C(\bar{D})$ of the *Dirichlet problem:*

$$\Delta u = 0 \quad \text{in } D, \tag{10}$$

$$u = f \quad \text{on } \partial D. \tag{11}$$

Here $C^2(D)$ denotes the set of real-valued functions that are defined and twice continuously differentiable on D. The set of real-valued functions that are defined and continuous on the closure $\bar{D}$ of D is denoted by $C(\bar{D})$. Physically, a solution of this Dirichlet problem yields the equilibrium temperature distribution in the region D when the temperature at the boundary of the region has a fixed distribution determined by the function f. It is well known that in order to solve this Dirichlet problem in general, one must impose a regularity condition on the boundary. Here a probabilistic definition of regularity is given, which is equivalent to the usual analytic one.

Definition. Let $\tau_D = \inf\{t > 0 : B(t) \notin D\}$. A point $x \in \partial D$ is *regular* if

$$P_x(\tau_D = 0) = 1.$$

The boundary, ∂D, is said to be regular if every point $x \in \partial D$ is regular.

Thus, a point $x \in \partial D$ is regular if and only if Brownian motion started at x hits $D^c \equiv \mathbb{R}^d \backslash D$ immediately after time zero, with probability one. There are examples of domains with boundary points that are not regular, Lebesgue's thorn being a classical example in three dimensions. The reader is referred to [5, p. 248] or [9, Section 7.10] for more details on this and on necessary and sufficient conditions for regularity of boundary points.

Theorem. *Suppose ∂D is regular. The following are equivalent.*

(*i*) $u \in C^2(D) \cap C(\bar{D})$ *satisfies* (10) and (11).
(*ii*) $u(x) = E_x[f(B(\tau_D))]$ *for all* $x \in \bar{D}$.

Proof. Itô's formula (8) will be used to prove that (i) implies (ii). That is, it will be used to give a probabilistic representation for solutions of the Dirichlet problem. Indeed, such probabilistic representations provide a convenient means for establishing uniqueness of solutions of partial differential equations. Given a function u satisfying (i), the representation (ii) actually holds for $x \in D$ without the assumption that ∂D is regular, as can be seen from the proof below. The converse, (ii) implies (i), requires more knowledge of the behavior of Brownian motion than is assumed here. A proof can be found, for instance, in [1, Chapter 4].

Let $\{D_n\}_{n=0}^{\infty}$ be a sequence of subdomains of D such that $\bar{D}_n \subset D_{n+1} \subset D$ for all n and $\bigcup_n D_n = D$. Fix $x \in D$ and let n be sufficiently large that $x \in D_n$. Since $u \in C^2(D)$ and $\bar{D}_n$ is compact, u can be extended off $\bar{D}_n$ to a function $u_n \in C_b^2(\mathbb{R}^d)$, the space of twice continuously differentiable functions that together with their first and second partial derivatives are bounded on $\mathbb{R}^d$. Applying (8) with $m = 0$ and $F = u_n$, one obtains P_x-a.s. for all $t \geq 0$:

$$u_n(B(t)) = u_n(B(0)) + \int_0^t \nabla u_n(B(s)) \cdot dB(s) + \frac{1}{2}\int_0^t \Delta u_n(B(s))ds. \quad (12)$$

The stochastic integral with respect to dB defines a martingale since ∇u_n is continuous and bounded on $\mathbb{R}^d$. By truncating time at $\tau_{D_n} \equiv \inf\{s \geq 0 : B(s) \notin D_n\}$, t and u_n can be replaced by $t \wedge \tau_{D_n}$ and u respectively in (12). By Doob's optional stopping theorem [1, p. 30], $\{\int_0^{t\wedge\tau_{D_n}} \nabla u(B(s)) \cdot dB(s), \mathcal{F}_t, t \geq 0\}$ is a martingale and hence has zero expectation under P_x. Also, $\triangle u = 0$ on D_n, so the last integral in (12) with $t \wedge \tau_{D_n}$ in place of t is 0. Hence, after replacing t by $t \wedge \tau_{D_n}$ in (12) and taking expectations there, one obtains

$$E_x\left[u(B(t \wedge \tau_{D_n}))\right] = u(x). \tag{13}$$

Since D is bounded, by the properties of Brownian motion (cf. [4, Ex. 12, p. 116]),

$$P_x(\tau_D < \infty) = 1, \tag{14}$$

which implies that $P_x(\tau_{D_n} < \infty) = 1$ for each n. Thus, by bounded convergence and the continuity of u, one can let $t \to \infty$ in (13) to obtain

$$E_x\left[u(B(\tau_{D_n}))\right] = u(x). \tag{15}$$

Now since $D_n^c \downarrow D^c$ and $x \in D$, $\tau_{D_n} \uparrow \tau_D$ P_x-a.s., and so by bounded convergence and the continuity of u on $\bar{D}$, on letting $n \to \infty$ in (15) one obtains

$$E_x\left[u(B(\tau_D))\right] = u(x) \quad \text{for all} \quad x \in D. \tag{16}$$

Since $u = f$ on ∂D, it follows that (ii) holds for x in D.

If $x \in \partial D$, then by the regularity of ∂D, $\tau_D = 0$ P_x-a.s., and then

$$E_x\left[f(B(\tau_D))\right] = f(x) = u(x). \qquad \square$$

3. SCHRÖDINGER EQUATION

Let D be a bounded domain in $\mathbb{R}^d$ with regular boundary, let q be a bounded, continuous (if $d = 1$) or Hölder continuous (if $d \geq 2$), real-valued function defined on D, and let f be a continuous real-valued function defined on the boundary ∂D of D. Define τ_D as in the previous section. For notational convenience, extend q to be zero off D. For each $t \geq 0$, define

$$e_q(t) = \exp\left(\int_0^t q(B(s))ds\right). \tag{17}$$

This $e_q(\cdot)$ is called the *Feynman-Kac* functional associated with q. It is a continuous one-dimensional process adapted to $\{\mathcal{F}_t\}$ and its sample paths are locally of bounded variation. Consider solutions $u \in C^2(D) \cap C(\bar{D})$ of the reduced *Schrödinger equation*:

$$\frac{1}{2}\Delta u + qu = 0 \text{ in } D, \tag{18}$$

$$u = f \quad \text{on } \partial D. \tag{19}$$

If q is non-positive, there is a simple representation for such solutions. This extends to positive q's, or q's which change sign, provided a certain integral (or gauge) condition (20) is satisfied.

Theorem. *Suppose*

$$\varphi(x) \equiv E_x[e_q(\tau_D)] < \infty \quad \text{for some} \quad x \in D. \tag{20}$$

The following are equivalent.

(i) $u \in C^2(D) \cap C(\bar{D})$ *satisfies (18) and (19).*
(ii) $u(x) = E_x[f(B(\tau_D))e_q(\tau_D)]$ *for all* $x \in \bar{D}$.

Proof. It is shown below that (i) implies (ii). The converse is more delicate, and in particular uses the assumption of continuity/Hölder continuity of q (see Chung [1, §4.7]).

Suppose (i) holds. Let $\{D_n\}_{n=0}^{\infty}$, $\{\tau_{D_n}\}_{n=0}^{\infty}$ be as in the previous section. Fix $x \in D$ and let n be sufficiently large that $x \in D_n$. As in Section 2, extend u off $\bar{D}_n$ to a function $u_n \in C_b^2(\mathbb{R}^d)$. Then by applying (8) with $m = 1$, $F(b, y) = u_n(b)\, y$ and $Y = e_q$, and then truncating time at τ_{D_n}, one obtains P_x-a.s. for all $t \geq 0$,

$$\begin{aligned} u(B(t \wedge \tau_{D_n}))\; e_q(t \wedge \tau_{D_n}) - u(B(0)) &= \int_0^{t\wedge\tau_{D_n}} e_q(s)\nabla u(B(s)) \cdot dB(s) \\ &+ \int_0^{t\wedge\tau_{D_n}} (qu)(B(s))e_q(s)ds \\ &+ \frac{1}{2}\int_0^{t\wedge\tau_{D_n}} \Delta u(B(s))e_q(s)ds. \end{aligned} \tag{21}$$

Since u satisfies (18) and $B(\cdot \wedge \tau_{D_n}) \in \bar{D}_n \subset D$ P_x-a.s., the sum of the last two integrals is zero. Now, $e_q \nabla u_n(B)$ is bounded on each compact time interval. It can be shown from this that $\{\int_0^t e_q(s)\nabla u_n(B(s)) \cdot dB(s), \mathcal{F}_t, t \geq 0\}$ is a martingale, and hence by Doob's optional stopping theorem, $\{\int_0^{t\wedge\tau_{D_n}} e_q(s)\nabla u(B(s)) \cdot dB(s), \mathcal{F}_t, t \geq 0\}$ is a martingale which has zero expectation. Thus taking expectations in (21) yields

$$E_x[u(B(t \wedge \tau_{D_n}))e_q(t \wedge \tau_{D_n})] = u(x). \tag{22}$$

Recall from Section 2 that $\tau_D < \infty$ P_x-a.s. It has been shown by Chung and Rao [2] (see also [4, §6.4]) that *under condition* (20), $\{u(B(t\wedge\tau_{D_n}))e_q(t\wedge\tau_{D_n}) :$

$t \geq 0,\ n \geq 0\}$ is uniformly integrable under P_x, and so, by combining this with the continuity of u on $\bar{D}$, one can pass to the limit as $t \to \infty$ and then $n \to \infty$ in (22) to obtain

$$E_x\left[u(B(\tau_D))e_q(\tau_D)\right] = u(x) \quad \text{for all} \quad x \in D.$$

Since $u = f$ on ∂D, this again reduces to (ii) for $x \in D$. If $x \in \partial D$, the regularity of ∂D gives the representation there. □

The function φ defined in (20) is known as the *Feynman-Kac gauge*. The following gives some equivalent conditions for finiteness of this gauge. For a proof, see [3]. (For $d = 1$, q is assumed to be Hölder continuous in [3]. However, scrutiny of the proof reveals that in this one-dimensional case the result still holds if q is simply bounded and continuous on D.)

Proposition. *The following conditions are equivalent.*

(*i*) $\varphi(x) < \infty$ *for some* $x \in D$.
(*ii*) *There is a solution* $u \in C^2(D) \cap C(\bar{D})$ of (18)–(19) *satisfying* $u > 0$ *on* $\bar{D}$.
(*iii*) *There is* no $\lambda \geq 0$ *such that the eigen-problem*

$$\begin{cases} \frac{1}{2}\Delta u + qu &= \lambda u \quad \text{in } D \\ \qquad\quad u &\equiv 0 \quad \text{on } \partial D \end{cases}$$

has a non-trivial solution $u \in C^2(D) \cap C(\bar{D})$.

Remark. If $q \leq 0$, then (i) is easily seen to hold and hence the Proposition yields that (ii)–(iii) hold, which is a well known fact in analysis. The case $q \equiv 0$ corresponds to the Dirichlet problem treated in Section 2.

4. REFLECTED BROWNIAN MOTION IN A QUADRANT

In this section, a process which is a functional of Brownian motion and that arises as an approximation to the queue-length process in a simple queueing network model will be considered. The queueing model is described first.

Consider two single-server queues in parallel (see Figure 1). The two arrival processes for these queues are assumed to be renewal processes that are independent of one another. The service times for a given server are assumed to form a sequence of independent, identically distributed random variables. The two sequences for the two servers are assumed to be independent of one another and of the arrival processes. The arrival rates are assumed to be equal and so are the service rates. Each server has an infinite waiting room. If the first server is ever idle, customers are transferred from the queue of the second server to that of the first server.

As it stands, in this generality, the queueing model cannot be analyzed exactly. However, with very general assumptions on the interarrival time and service time distributions and service disciplines, under conditions of *heavy traffic* (mean interarrival times roughly equal to mean service times), one can approximate the two-dimensional queue length process for this pair of queues by a diffusion process that lives in the positive quadrant of $\mathbb{R}^2$ [11]. This diffusion behaves like Brownian motion in the interior of the quadrant, and it is confined to the quadrant by instantaneous "pushing" at the boundary, where the direction of push is constant on a given side and these directions are illustrated in Figure 2. For historical reasons the directions of push are called *directions of reflection,* although one should not think of the construction as by a mirror reflection, but rather by "deflection" or "pushing" at the boundary in the prescribed directions. The directions of reflection have a natural interpretation in terms of the original queueing model. Namely, the normal reflection on the horizontal boundary corresponds to the enforcement of the non-negativity constraint on the contents of queue 2, whereas the 45° downward reflection on the vertical boundary corresponds to the enforcement of the non-negativity constraint on queue 1 as well as the fact that, when server 1 has no customers from his queue to serve, he can serve customers from queue 2, causing a corresponding decrement in the contents of queue 2.

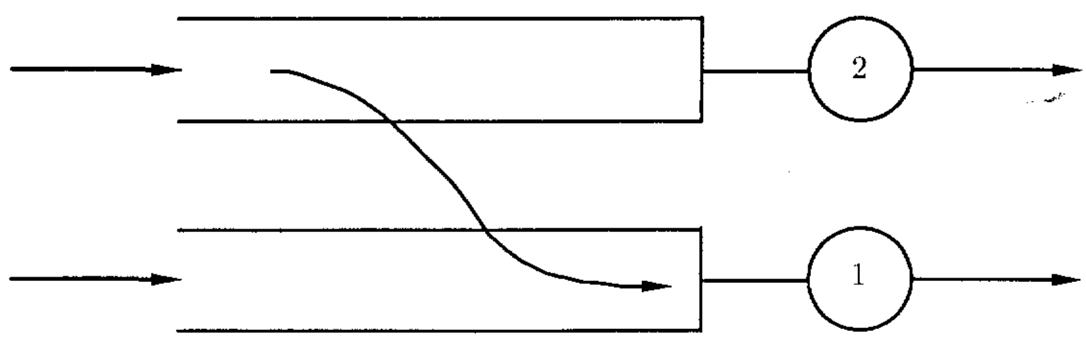

FIGURE 1.

In fact, there is an explicit representation for the diffusion process described above. Let B be a two-dimensional Brownian motion starting from some point x in the positive quadrant $\mathbb{R}^2_+$. Define for each $t \geq 0$,

$$\begin{aligned}
Y_1(t) &= (-\min_{0\leq s\leq t} B_1(s))^+ \\
Z_1(t) &= B_1(t) + Y_1(t) \\
Y_2(t) &= \left(-\min_{0\leq s\leq t} (B_2(s) - Y_1(s))\right)^+ \\
Z_2(t) &= B_2(t) - Y_1(t) + Y_2(t),
\end{aligned}$$

where $y^+ = \max(y, 0)$ for $y \in \mathbb{R}$. Then $Z \equiv \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}$ is a reflected Brownian motion in $\mathbb{R}^2_+$ that starts from x and has directions of reflection as indicated in

Figure 2. Indeed, $Y_1,\ Y_2$ may be characterized as the unique pair of continuous, non-decreasing processes such that

(*i*) $Z(t) \equiv B(t) + v_1 Y_1(t) + v_2 Y_2(t) \in \mathbb{R}^2_+$ for all $t \geq 0$, and

(*ii*) $Y_i(0) = 0$, Y_i can increase only when Z_i is zero, $i = 1, 2$,

where $v_1 = \binom{1}{-1}$, $v_2 = \binom{0}{1}$ are the directions of reflection (normalized to have inward normal component of length one) shown in Figure 2.

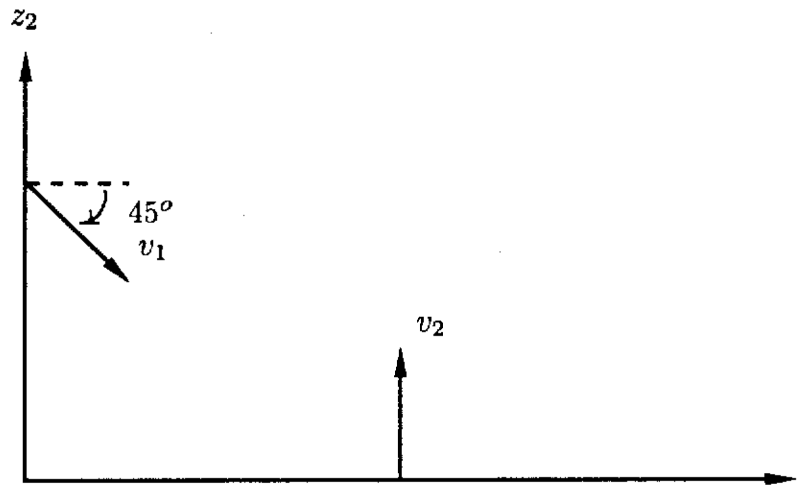

FIGURE 2.

One of the questions of interest for the process Z is whether it ever reaches the origin starting from $x \neq 0$. The following function, together with Itô's formula, allows us to answer this question in a precise manner. Let (r, θ) be polar coordinates in $\mathbb{R}^2_+$ with $r \geq 0$ and $\theta \in [0,\ \frac{\pi}{2}]$. Let

$$u(r, \theta) = r^{\frac{1}{2}} \cos\left(\frac{1}{2}\theta\right). \tag{23}$$

Then u is the real part of the complex function $z^{\frac{1}{2}}$ and consequently is harmonic in $\mathbb{R}^2_+ \backslash \{0\}$. Moreover, for $r > 0$,

$$\frac{1}{r}\frac{\partial u}{\partial \theta}\bigg|_{\theta=0} = -\frac{1}{2} r^{-\frac{1}{2}} \sin\left(\frac{1}{2}\theta\right)\bigg|_{\theta=0} = 0$$

and

$$\left(-\frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{\partial u}{\partial r}\right)\bigg|_{\theta=\frac{\pi}{2}} = \frac{1}{2}\left(r^{-\frac{1}{2}} \sin\left(\frac{1}{2}\theta\right) - r^{-\frac{1}{2}} \cos\left(\frac{1}{2}\theta\right)\right)\bigg|_{\theta=\frac{\pi}{2}} = 0.$$

Interpreting these equations in Cartesian coordinates yields

$$v_2 \cdot \nabla u = 0 \quad \text{on} \quad \{z \in \mathbb{R}^2_+ \backslash \{0\} :\ z_2 = 0\}, \tag{24}$$

$$v_1 \cdot \nabla u = 0 \quad \text{on} \quad \{z \in \mathbb{R}^2_+ \setminus \{0\} : \ z_1 = 0\}. \tag{25}$$

Suppose $x \in \mathbb{R}^2_+ \setminus \{0\}$ and fix ϵ, R such that $0 < \epsilon < u(x) < R < \infty$. Let $\bar{D}_{\epsilon R} = \{z \in \mathbb{R}^2_+ : \epsilon \leq u(z) \leq R\}$. Then one can extend u off $\bar{D}_{\epsilon R}$ to a function $\tilde{u} \in C^2_b(\mathbb{R}^2)$. By applying Itô's formula (8) with $F : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ given by $F(b, y) = \tilde{u}(b + v_1 y_1 + v_2 y_2)$ for $b \in \mathbb{R}^2$, $y \in \mathbb{R}^2$, one obtains P_x-a.s. for all $t \geq 0$,

$$\begin{aligned} \tilde{u}(Z(t)) = \tilde{u}(Z(0)) \quad &+ \quad \int_0^t \nabla \tilde{u}(Z(s)) \cdot dB(s) \\ &+ \quad \sum_{i=1}^2 \int_0^t (v_i \cdot \nabla \tilde{u})(Z(s)) dY_i(s) \\ &+ \quad \frac{1}{2} \int_0^t \Delta \tilde{u}(Z(s)) ds. \end{aligned} \tag{26}$$

The stochastic integral with respect to dB defines a martingale since $\nabla \tilde{u}$ is continuous and bounded on $\mathbb{R}^2$. By truncating time at $\tau_{\epsilon R} \equiv \inf\{t \geq 0 : u(Z(t)) \leq \epsilon \text{ or } u(Z(t)) \geq R\}$, and noting that $\Delta u = 0$ in $\bar{D}_{\epsilon R}$, one obtains P_x-a.s. for all $t \geq 0$,

$$\begin{aligned} u(Z(t \wedge \tau_{\epsilon R})) = u(x) \quad &+ \quad \int_0^{t \wedge \tau_{\epsilon R}} \nabla u(Z(s)) \cdot dB(s) \\ &+ \quad \sum_{i=1}^2 \int_0^{t \wedge \tau_{\epsilon R}} (v_i \cdot \nabla u)(Z(s)) dY_i(s), \end{aligned} \tag{27}$$

where by Doob's optional stopping theorem the first integral defines a martingale, which has zero expectation. Moreover, since Y_i can increase only when $Z_i = 0$, and $v_i \cdot \nabla u = 0$ on $\{z \in \mathbb{R}^2_+ \setminus \{0\} : z_i = 0\}$, the integrals in the last term of (27) are zero. Thus, taking expectations in (27) yields

$$E_x \left[u(Z(t \wedge \tau_{\epsilon R})) \right] = u(x). \tag{28}$$

Now, one would like to let $t \to \infty$ in (28). For this, the following is needed.

Proposition. *For each* $K > 0$,

$$P_x(\sigma_K < \infty) = 1,$$

where $\sigma_K = \inf\{t \geq 0 : |Z(t)| \geq K\}$.

Proof. From (i), for $v = \binom{1}{1}$,

$$v \cdot Z(t) = v \cdot B(t) + Y_2(t) \geq v \cdot B(t) \quad \text{for all } t \geq 0, \tag{29}$$

since $Y_2(t) \geq 0$. Now, $v \cdot B$ is equal in distribution to $\sqrt{2}$ times a one-dimensional Brownian motion and consequently

$$P_x(\limsup_{t\to\infty} v \cdot B(t) = +\infty) = 1,$$

and hence by (29),

$$P_x(\limsup_{t\to\infty} v \cdot Z(t) = +\infty) = 1.$$

Since $v \cdot Z \leq 2|Z|$, the Proposition follows. □

Now, $u(z) \geq |z|^{\frac{1}{2}} c$ where $c = \inf_{\theta\in[0,\frac{\pi}{2}]} \cos(\frac{1}{2}\theta) > 0$, and so it follows from the above Proposition that

$$P_x(\tau_{\epsilon R} < \infty) = 1. \tag{30}$$

Hence, since u is bounded on $\bar{D}_{\epsilon R}$, one can let $t \to \infty$ in (28) to obtain by bounded convergence that

$$E_x\left[u(Z(\tau_{\epsilon R}))\right] = u(x). \tag{31}$$

By observing the values of u at $Z(\tau_{\epsilon R})$, one concludes from (31) that

$$\epsilon P_x(\tau_\epsilon < \tau_R) + R P_x(\tau_R < \tau_\epsilon) = u(x), \tag{32}$$

where $\tau_r = \inf\{t \geq 0 : u(Z(t)) = r\}$. Since $P_x(\tau_R < \tau_\epsilon) = 1 - P_x(\tau_\epsilon < \tau_R)$ by (30), one can rearrange (32) to obtain

$$P_x(\tau_\epsilon < \tau_R) = \frac{R - u(x)}{R - \epsilon}.$$

On letting $\epsilon \downarrow 0$ one obtains

$$P_x(\tau_0 < \tau_R) = 1 - \frac{u(x)}{R}.$$

Finally, letting $R \to \infty$ yields

$$P_x(\tau_0 < \infty) = 1.$$

Thus, the reflected Brownian motion Z hits the origin P_x-a.s. starting from any $x \in \mathbb{R}^2_+ \backslash \{0\}$.

Reflected Brownian motions in two-dimensional polygons and in three and higher dimensional polyhedrons, with constant oblique reflection on each boundary face, arise as approximations to other queueing network models under conditions of heavy traffic. For more discussion of such processes, the interested reader is referred to [6, 7, 8, 11, 12, 13].

Bibliographical note. The reader interested in pursuing more details and applications of stochastic calculus is referred to the books [4, 5], and for further details on many aspects of Brownian motion, see [1, 9, 10].

References

[1] **Chung, K. L.,** *Lectures from Markov Processes to Brownian Motion,* Springer-Verlag, New York, 1982.

[2] **Chung, K. L. and Rao, K. M.,** Feynman-Kac functional and the Schrödinger equation, *Seminar on Stochastic Processes 1981,* eds. E. Çinlar, K.L. Chung, R.K. Getoor, Birkhäuser, Boston, 1981, 1–29.

[3] **Chung, K. L., Li P. and Williams, R. J.,** Comparison of probability and classical methods for the Schrödinger equation, *Expositiones Mathematica,* **4**, 271–278, 1986.

[4] **Chung, K. L. and Williams, R. J.,** *Introduction to Stochastic Integration,* 2nd Edition, Birkhäuser, Boston, 1990.

[5] **Durrett, R.,** *Brownian Motion and Martingales in Analysis,* Wadsworth, Belmont, CA, 1984.

[6] **Harrison, J. M.,** *Brownian Motion and Stochastic Flow Systems,* John Wiley & Sons, New York, 1985.

[7] **Harrison, J. M. and Williams, R. J.,** Brownian models of open queueing networks with homogeneous customer populations, *Stochastics,* **22**, 77–115, 1987.

[8] **Harrison, J. M. and Williams, R. J.,** Brownian models of multiclass queueing networks, *Proc. 29th I.E.E.E. Conf. on Decision and Control,* 1990, 573-574.

[9] **Itô, K. and McKean, H. P. Jr.,** *Diffusion Processes and their Sample Paths,* Springer-Verlag, New York, 1974.

[10] **Karatzas, I. and Shreve, S. E.,** *Brownian Motion and Stochastic Calculus,* Springer-Verlag, 1988.

[11] **Reiman, M. I.,** Open queueing networks in heavy traffic, *Math. Oper. Res.,* **9**, 441–458, 1984.

[12] **Taylor, L. M. and Williams, R. J.,** Existence and uniqueness of semimartingale reflecting Brownian motions in an orthant, to appear in *Probability Theory and Related Fields*, 1993–1994.

[13] **Varadhan, S. R. S. and Williams, R. J.,** Brownian motion in a wedge with oblique reflection, *Comm. Pure Appl. Math.*, **38**, 405–443, 1985.