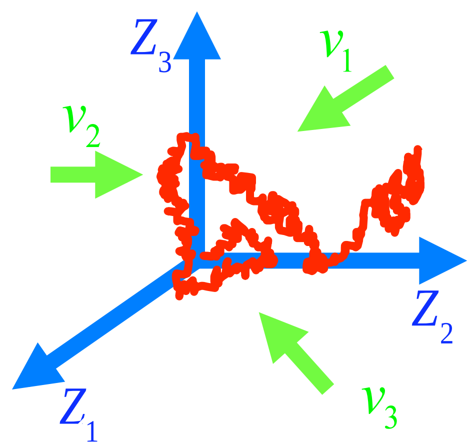
A RANDOM WALK THROUGH ANALYSIS, NETWORKS AND BIOLOGY
Lecture 2



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G C Steward Visiting Fellow
Gonville and Caius College
April 2010

CONNECTIONS

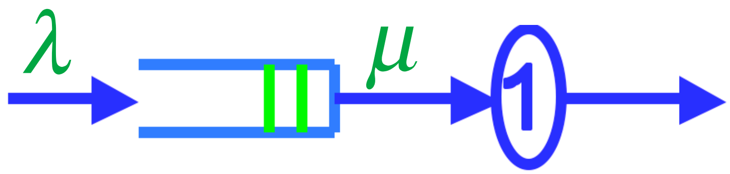
- Brownian motion and analysis
- Reflecting Brownian motion and queueing networks
- Queues and biology

QUEUES

Photograph courtesy of Ilze Ziedins

SINGLE SERVER QUEUE

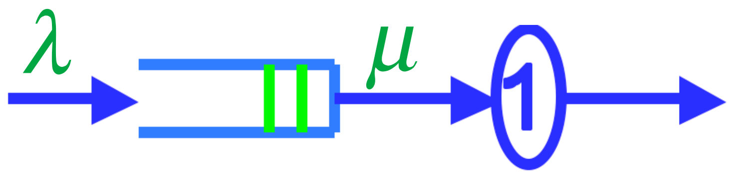
M/M/1 Queue



- Poisson arrivals at rate λ
- i.i.d. exponential service times mean μ^{-1}

Kendall 1918-2007

M/M/1 Queue



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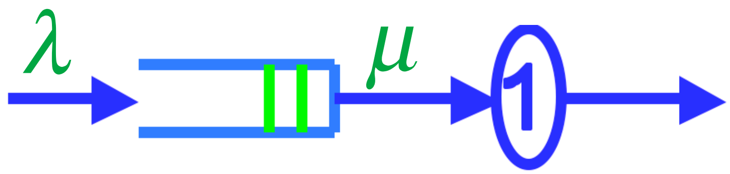
- FIFO order of service, infinite buffer

M/M/1 Queue

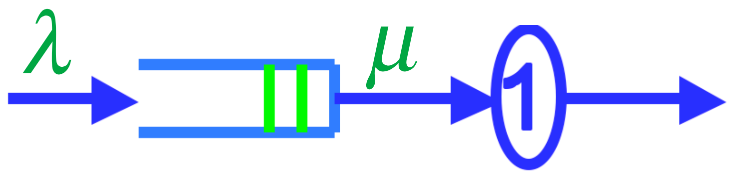


- Poisson arrivals at rate λ
- i.i.d. exponential service times mean μ^{-1}
- FIFO order of service, infinite buffer
- ***Q(t)***=queuelength at time *t* (includes customer being served)

M/M/1 queue
(Java simulation applet)

- http://homepages.inf.ed.ac.uk/jeh/Simjava/queueing/mm1_q/mm1_q.html

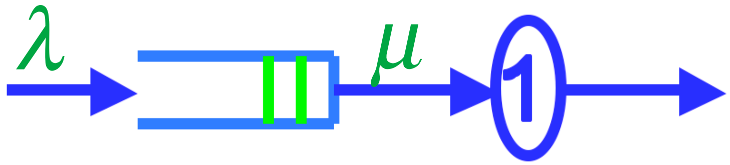
Balanced M/M/1 Queue



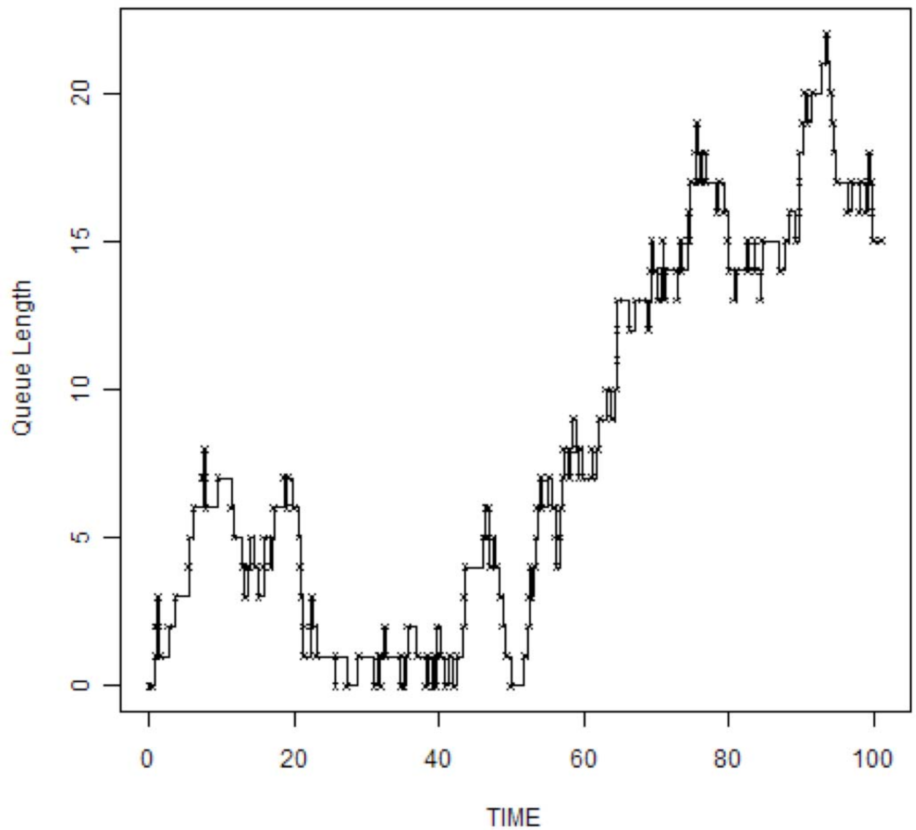
- Poisson arrivals at rate λ
- i.i.d. exponential service times mean μ^{-1}
- FIFO order of service, infinite buffer
- ***Q*(*t*)**=queuelength at time *t* (includes customer being served)

- Balanced $\lambda = \mu$ (heavy traffic)

Balanced M/M/1 Queue
(Simulation)



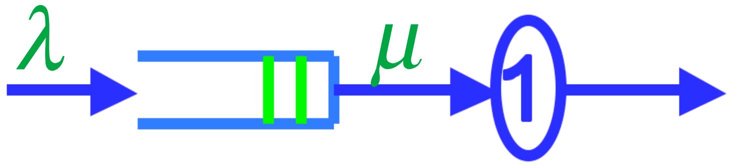
$\lambda = 1, \quad \mu = 1$



Simulations in R courtesy of Nam H. Lee

Balanced M/M/1 Queue

(Simulation)



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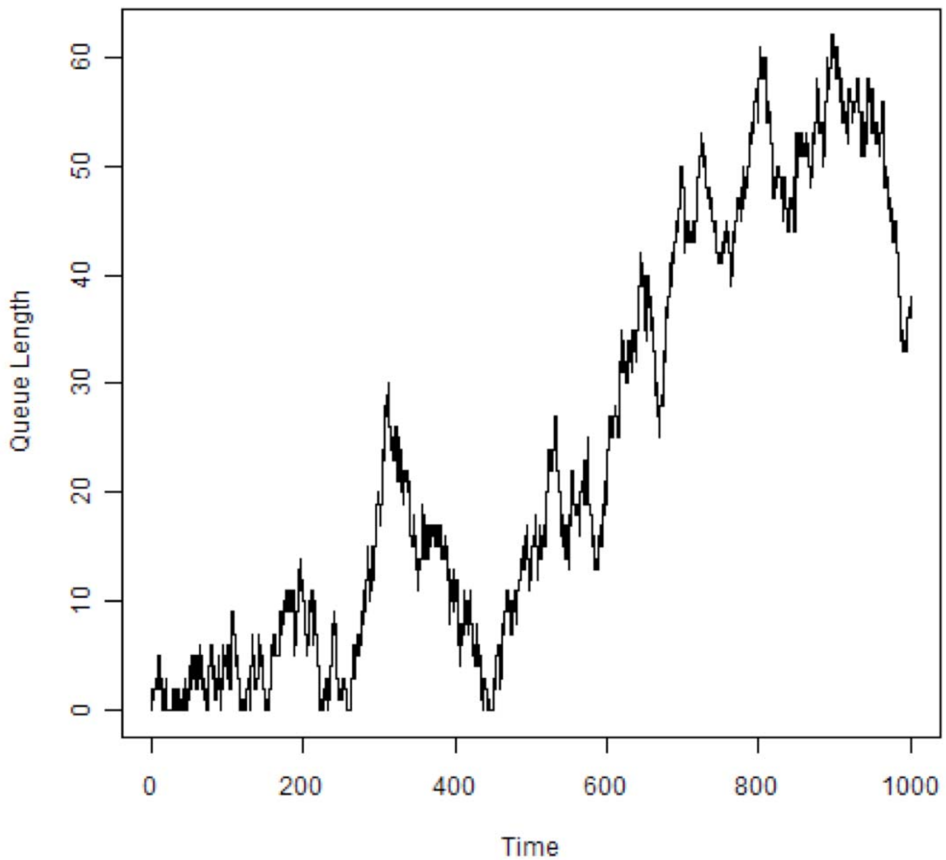

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Balanced M/M/1 Queue
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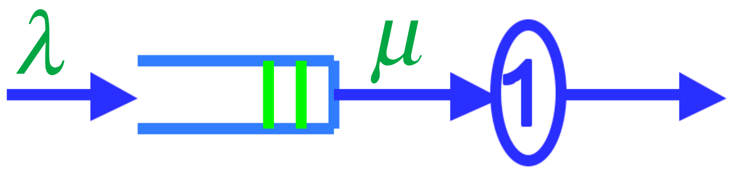

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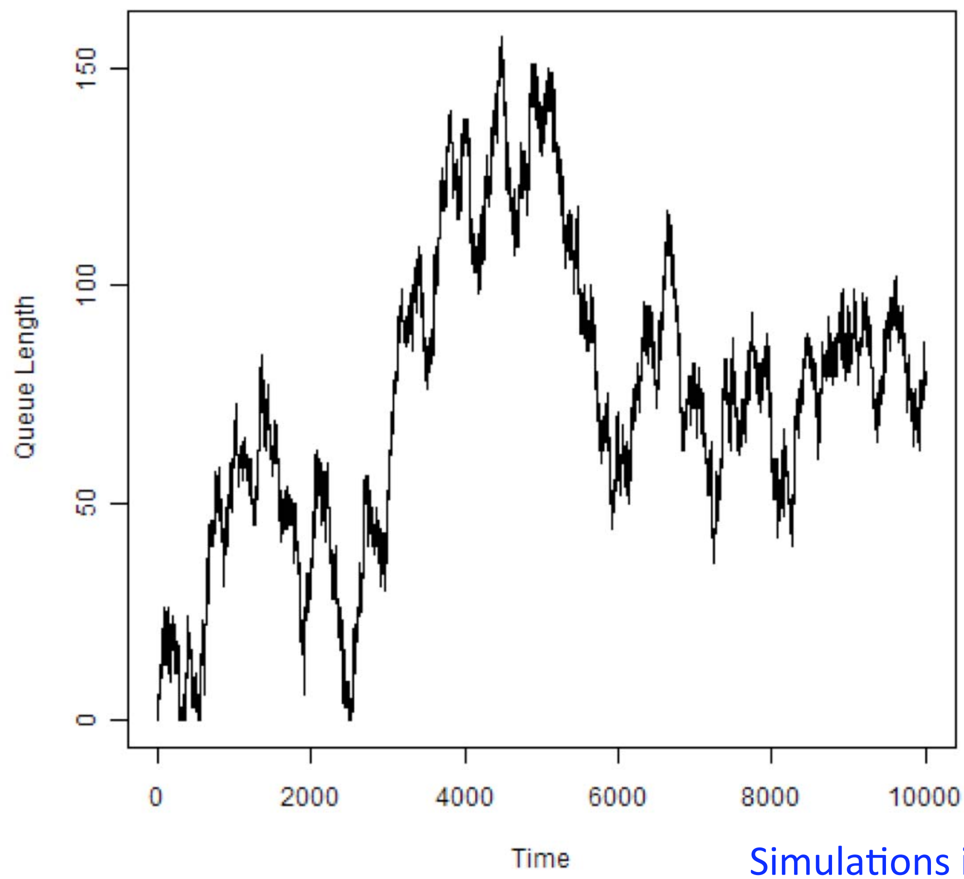
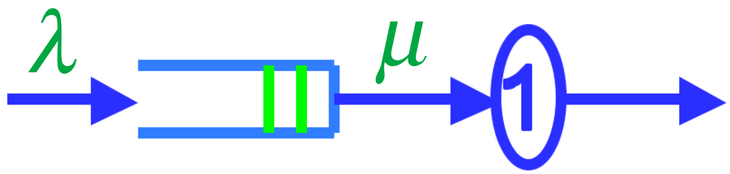


Simulations in R courtesy of Nam H. Lee

Balanced M/M/1 Queue

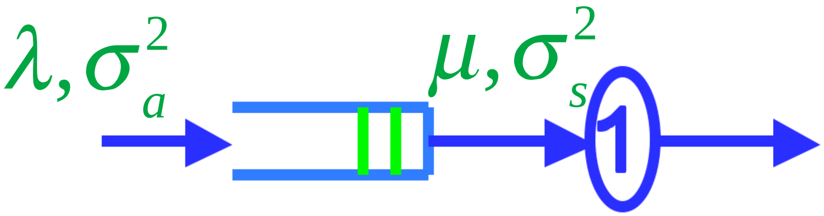


$$\lambda = \mu$$

$$\hat{Q}^m(\cdot) \triangleq \frac{1}{\sqrt{m}} Q(m\cdot) \Rightarrow Z \quad \text{as } m \to \infty$$

where ***Z*** is a reflecting Brownian motion - RBM (with variance parameter 2λ)

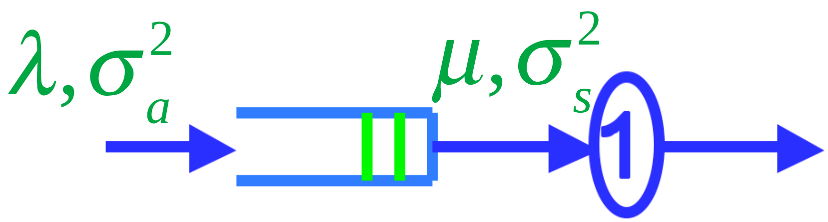
Balanced GI/GI/1 Queue



$$\lambda = \mu$$

i.i.d. interarrival times and i.i.d. service times

Balanced GI/GI/1 Queue



$$\lambda = \mu$$

i.i.d. interarrival times and i.i.d. service times

<u>Theorem (A. Borovkov '67, Iglehart-Whitt '70)</u>

$$\hat{Q}^m(\cdot) \triangleq \frac{1}{\sqrt{m}} Q(m\cdot) \Rightarrow Z \quad \text{as } m \to \infty$$

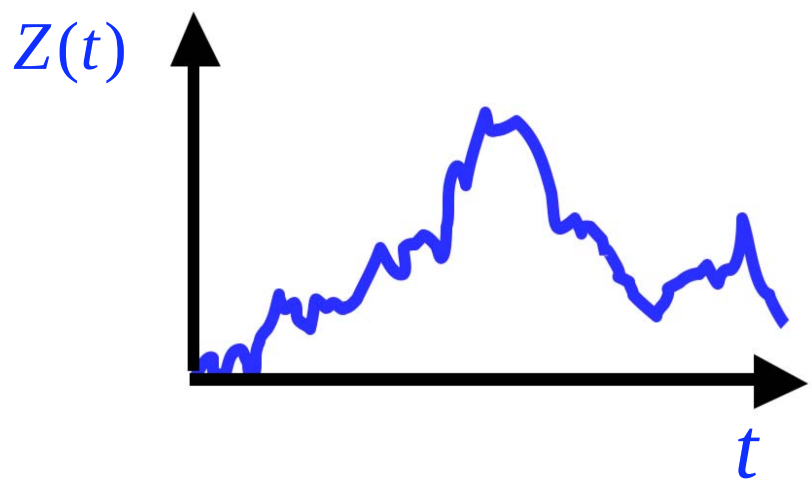
where ***Z*** is a reflecting Brownian motion with variance parameter $\lambda^3(\sigma_a^2 + \sigma_s^2)$

Reflecting Brownian Motion

$$Z(t) = B(t) + Y(t)$$

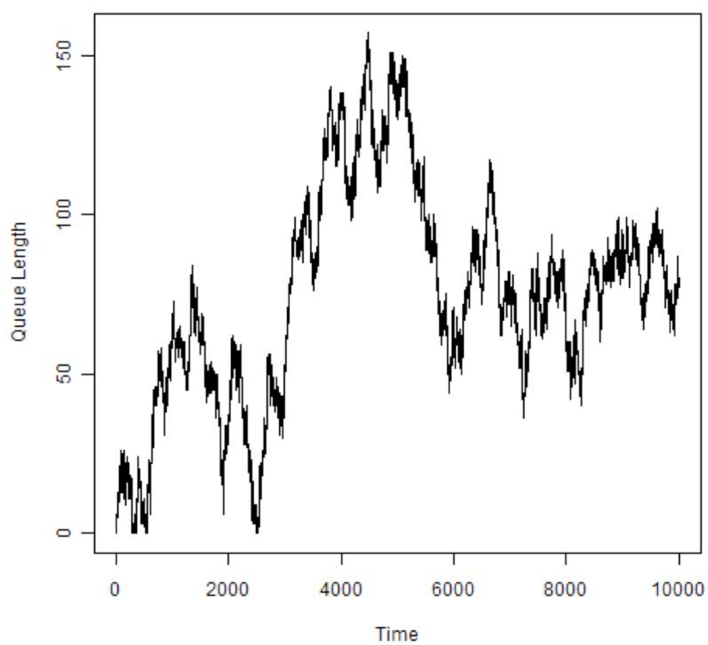
Brownian motion

$$Y(t) = \max\{-B(s) : 0 \le s \le t\}$$



Law of the Iterated Logarithm

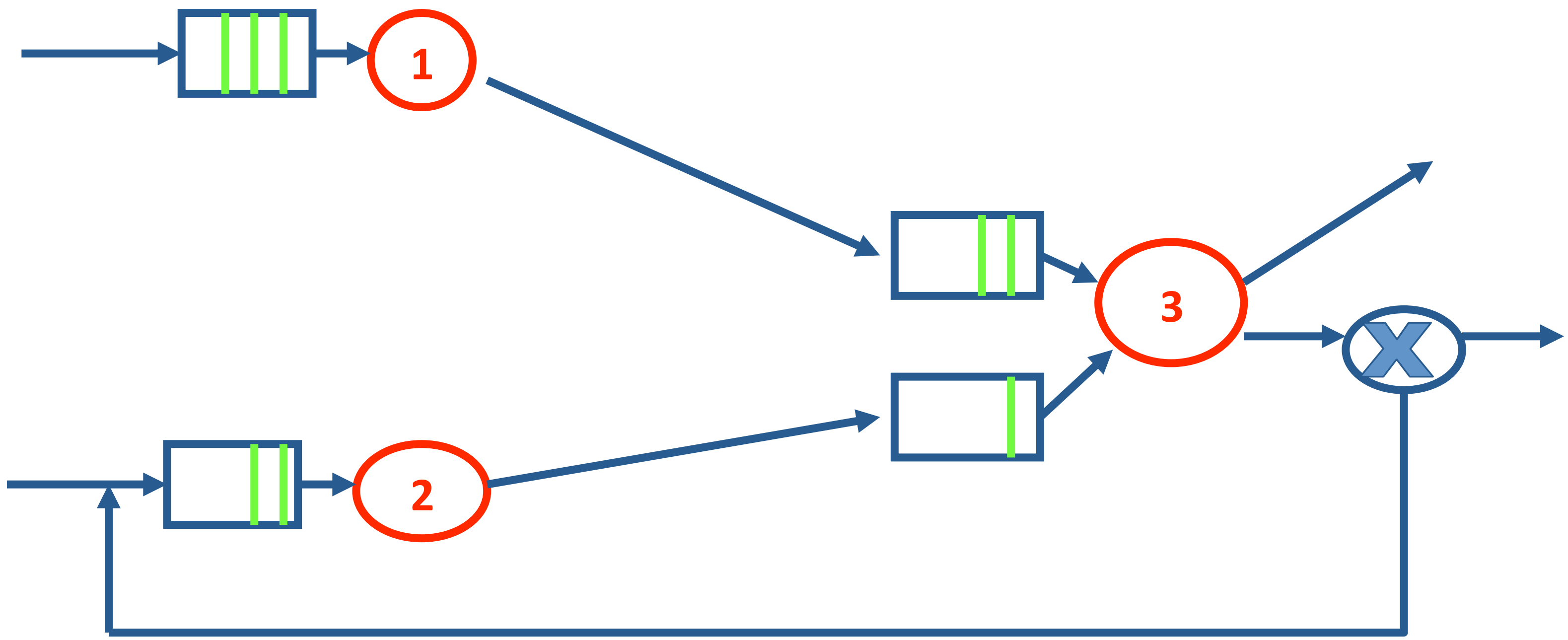
$$\limsup_{t\to\infty} \frac{Z(t)}{\sqrt{t \log\log t}} = 2\sqrt{\lambda} \quad \text{a.s.}$$



QUEUEING NETWORKS

QUEUEING NETWORKS

- Applications to manufacturing, telecommunications, computer systems, service networks



TANDEM QUEUE

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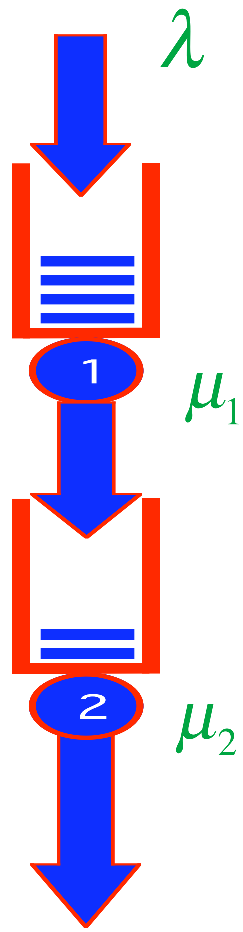
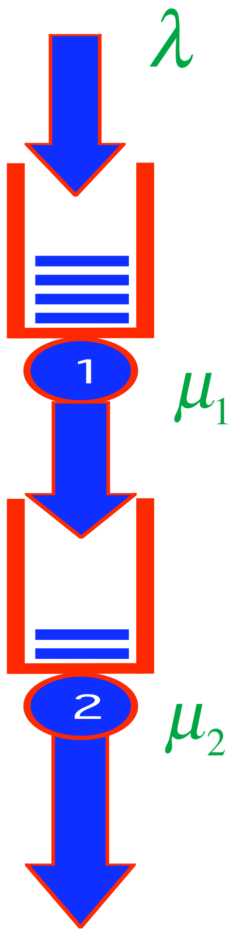


TANDEM QUEUE

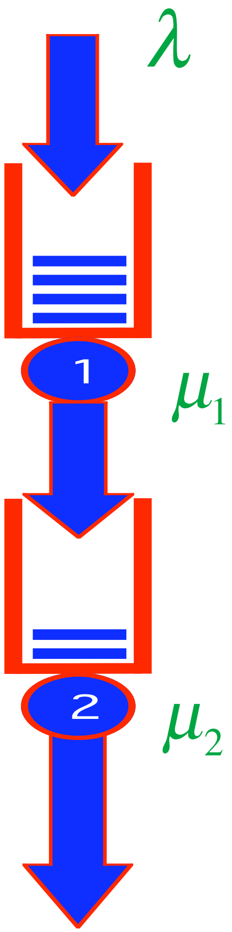


i.i.d. interarrival times, mean $1/\lambda$

i.i.d. service times, mean $1/\mu_1$

i.i.d. service times, mean $1/\mu_2$

BALANCED TANDEM QUEUE



Balance condition (heavy traffic)

$$\lambda = \mu_1 = \mu_2$$

Queuelength process

$Q_i(t) =$ length of queue at station i at time t

BALANCED TANDEM QUEUE

$$\hat{Q}^m(\cdot) \triangleq \frac{1}{\sqrt{m}} Q(m\cdot) \Rightarrow Z \quad \text{as } m \to \infty$$

where Z is a two-dimensional reflecting Brownian motion **(Iglehart & Whitt `70)**

BALANCED TANDEM QUEUE

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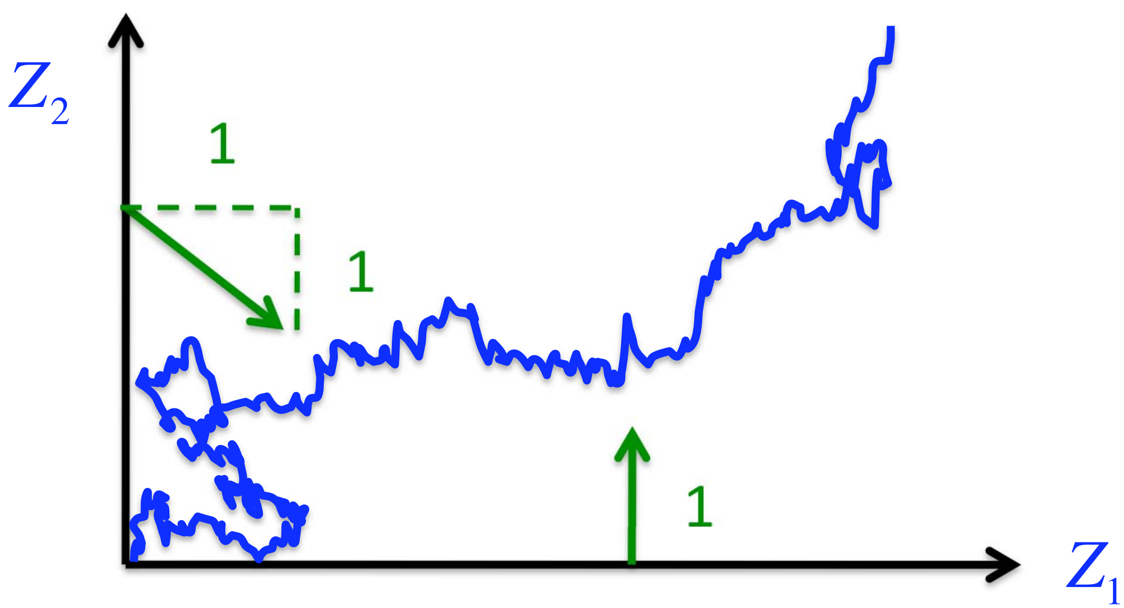

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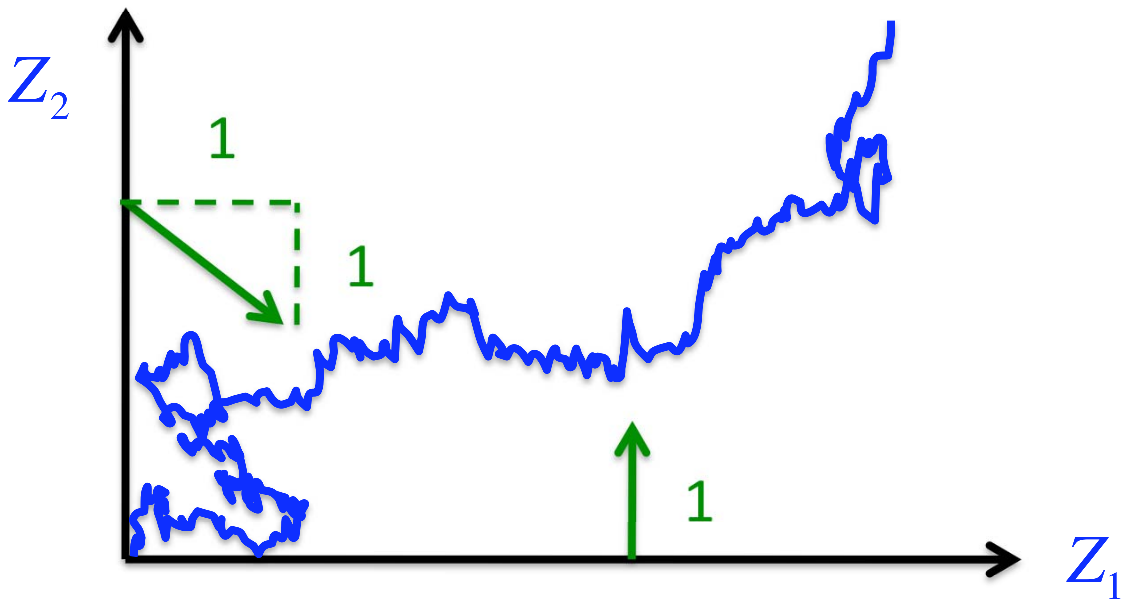
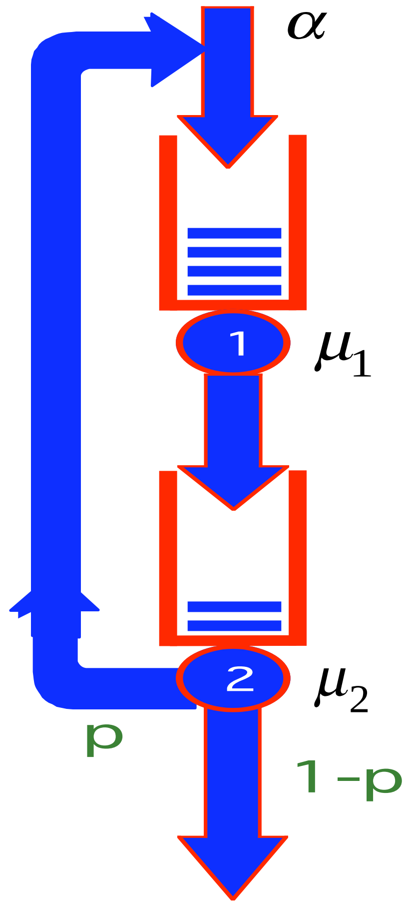


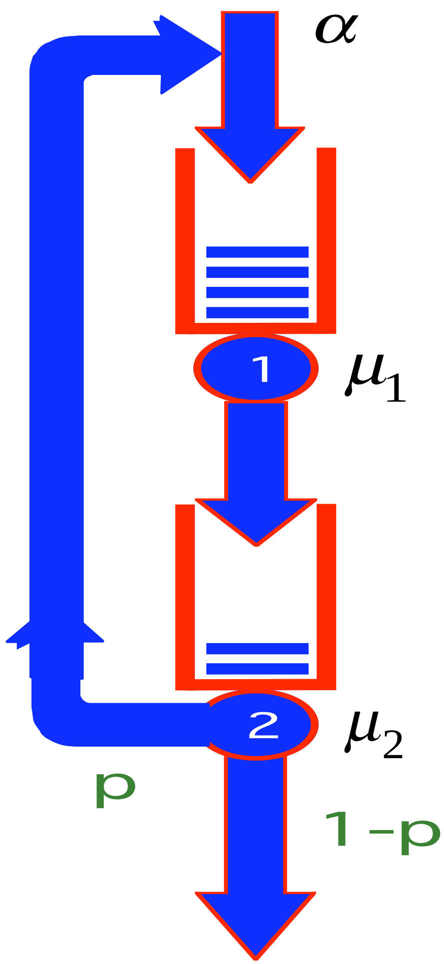
$$Z = B + RY$$

$$R = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

TANDEM QUEUE WITH FEEDBACK



TANDEM QUEUE WITH FEEDBACK



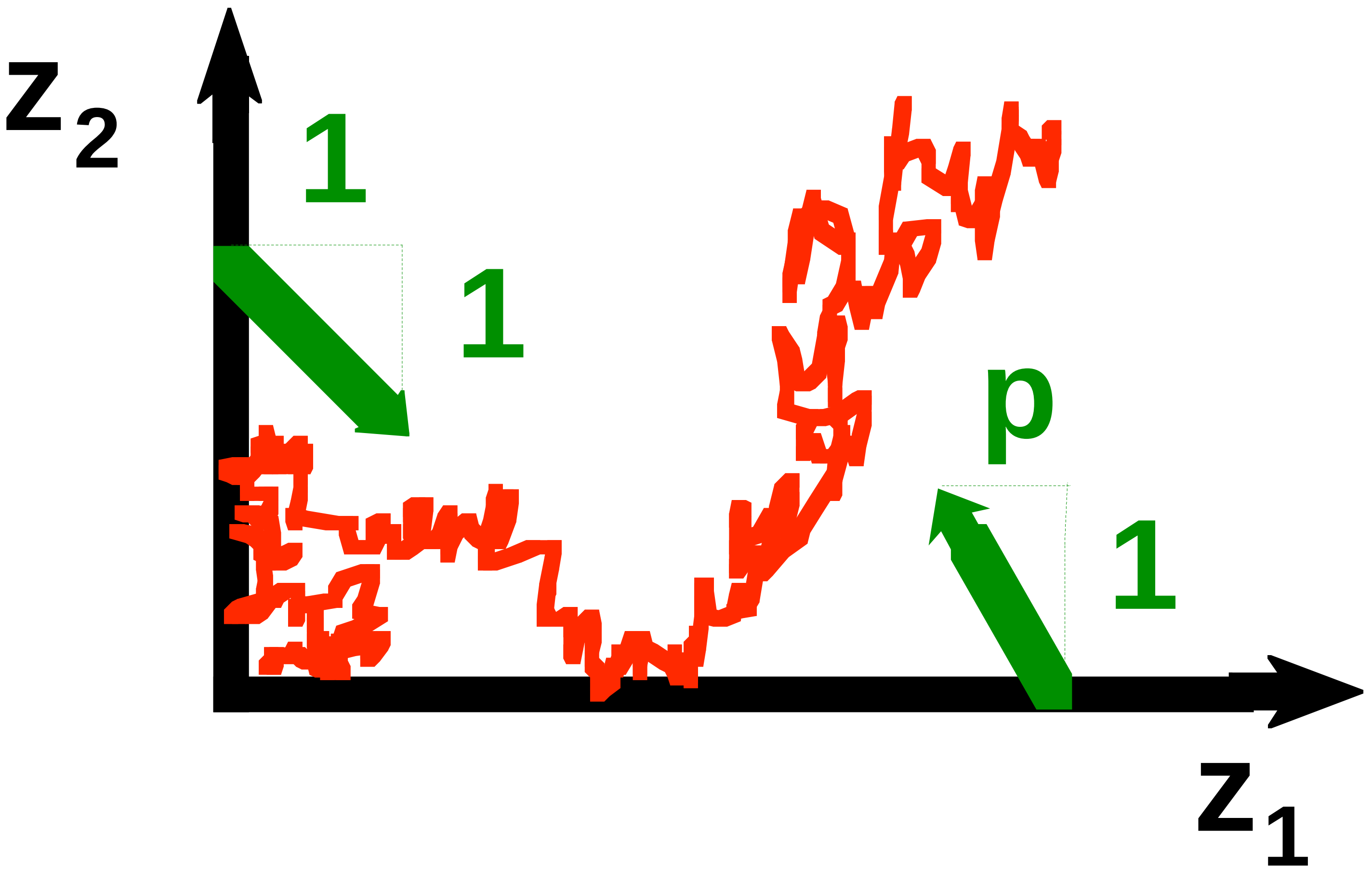
Effective arrival rate to station 1

$$\lambda = \alpha + p\lambda$$

Balance condition (heavy traffic)

$$\lambda = \mu_1 = \mu_2$$

TWO-DIMENSIONAL REFLECTING BROWNIAN MOTION



TWO-DIMENSIONAL REFLECTING BROWNIAN MOTION

Some questions:

-Does *Z* hit the origin?

-For more general directions of reflection, does *Z* escape from the origin (uniquely)?

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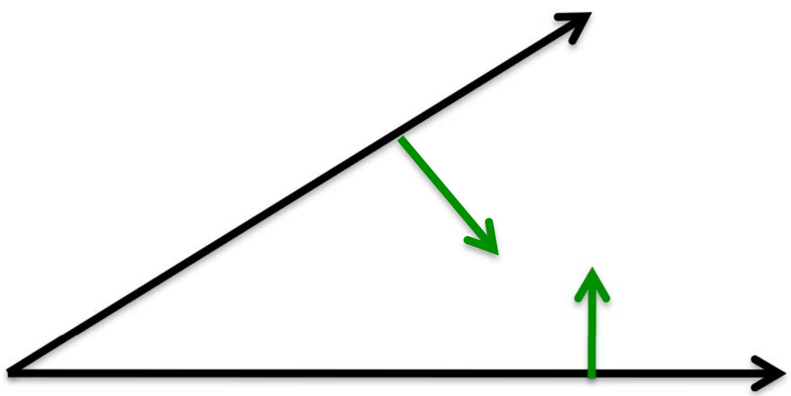
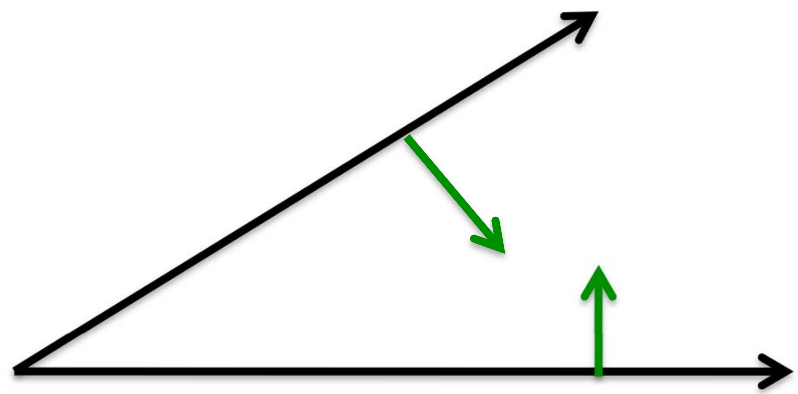


RBM HITS THE CORNER?



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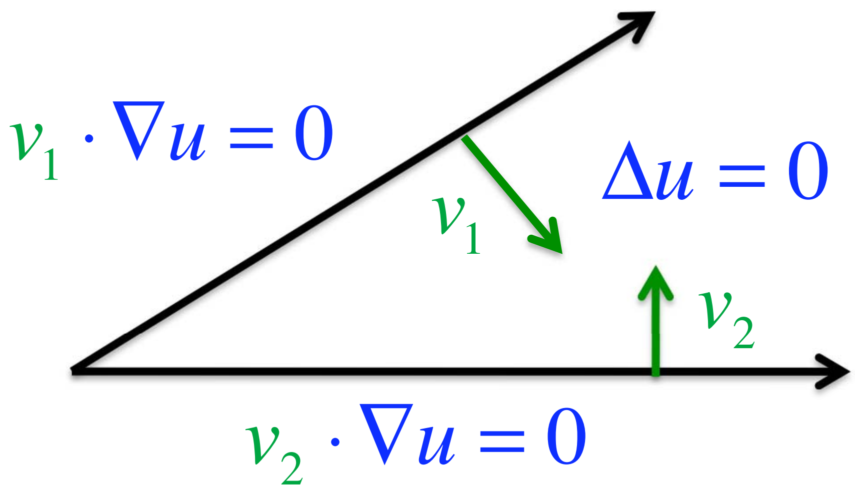

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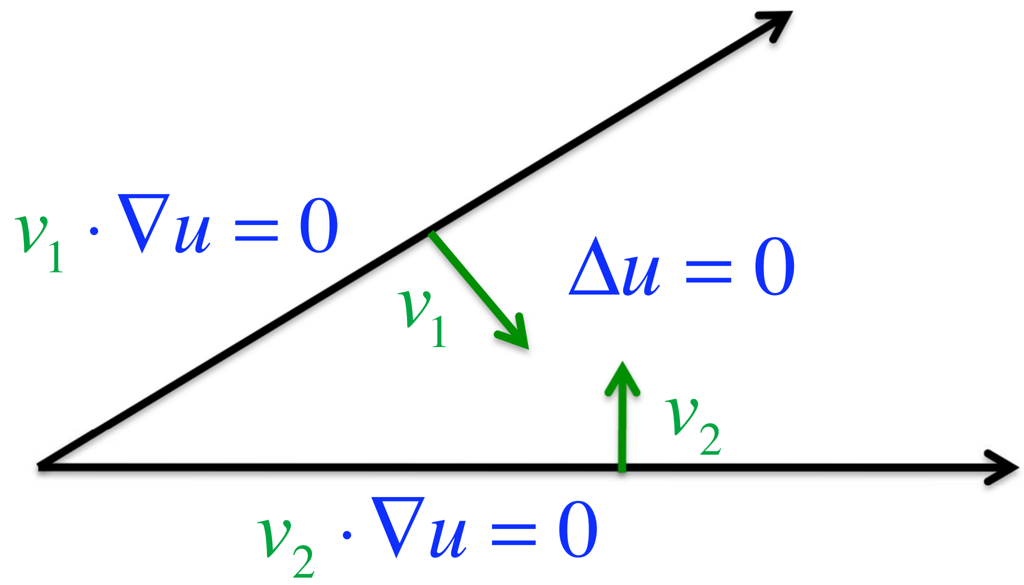
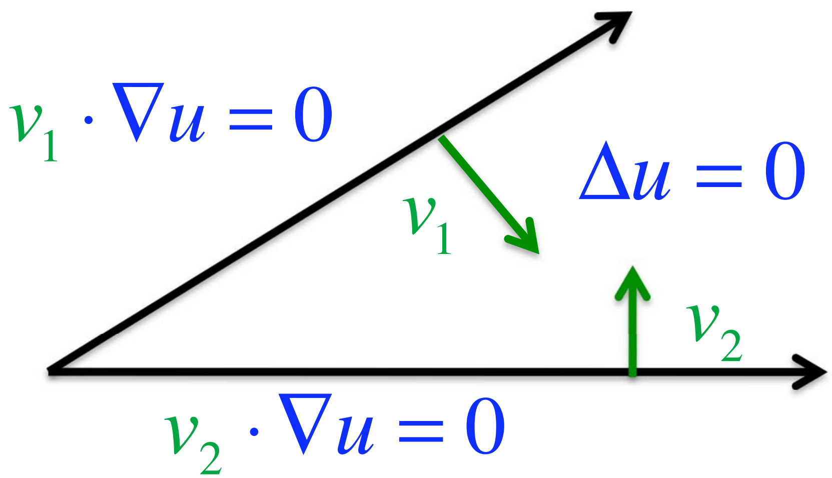


$$u(r,\theta) = r^{\alpha} \cos(\alpha\theta - \theta_1) \qquad \alpha = (\theta_1 + \theta_2) / \xi$$

RBM HITS THE CORNER?

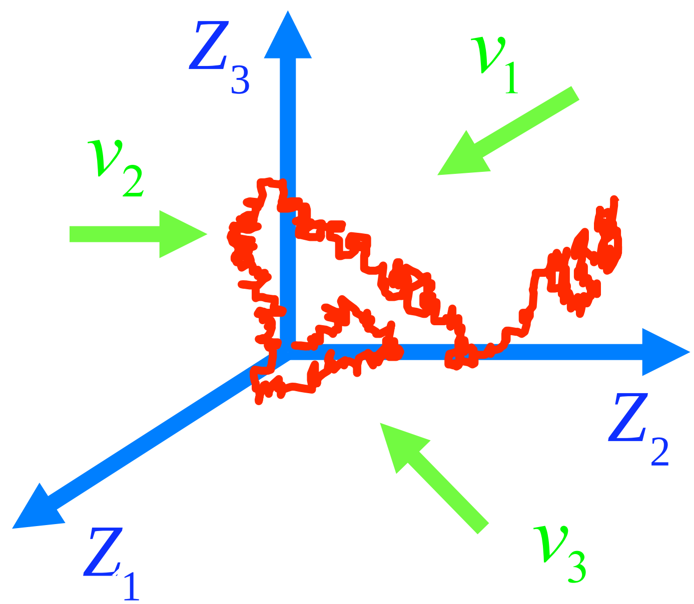


$$u(r,\theta) = r^\alpha \cos(\alpha\theta - \theta_1) \qquad \alpha = (\theta_1 + \theta_2)/\xi$$

Hit the corner with probability one or zero according to whether $\alpha > 0$ or $\alpha \le 0$

MULTIDIMENSIONAL REFLECTING BROWNIAN MOTION

- In higher dimensions, is *Z* well defined?
- How does it behave?



CONNECTIONS

- **Brownian motion and analysis**
- **Reflecting Brownian motion and queuing networks**
- **Queues and biology**

- **Webpage:**

http://www.math.ucsd.edu/~williams/talks/caius/gcsteward2010.html

THANK YOU