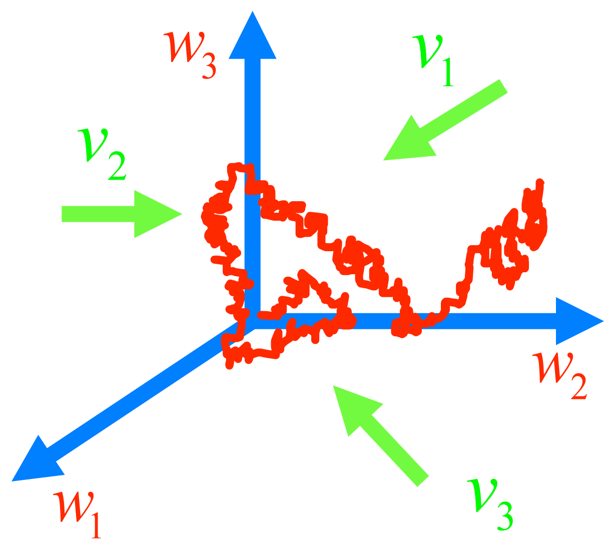
A RANDOM WALK THROUGH ANALYSIS, NETWORKS AND BIOLOGY

In this series of three lectures, I will describe some connections between probability and other fields. I will begin by introducing the fundamental stochastic process of Brownian motion and will illustrate some connections to partial differential equations via Ito's stochastic calculus. I will then introduce a variant called reflecting Brownian motion which arises in applications to queueing networks. Finally, I will illustrate a connection between the probability theory of queues and synthetic biology.

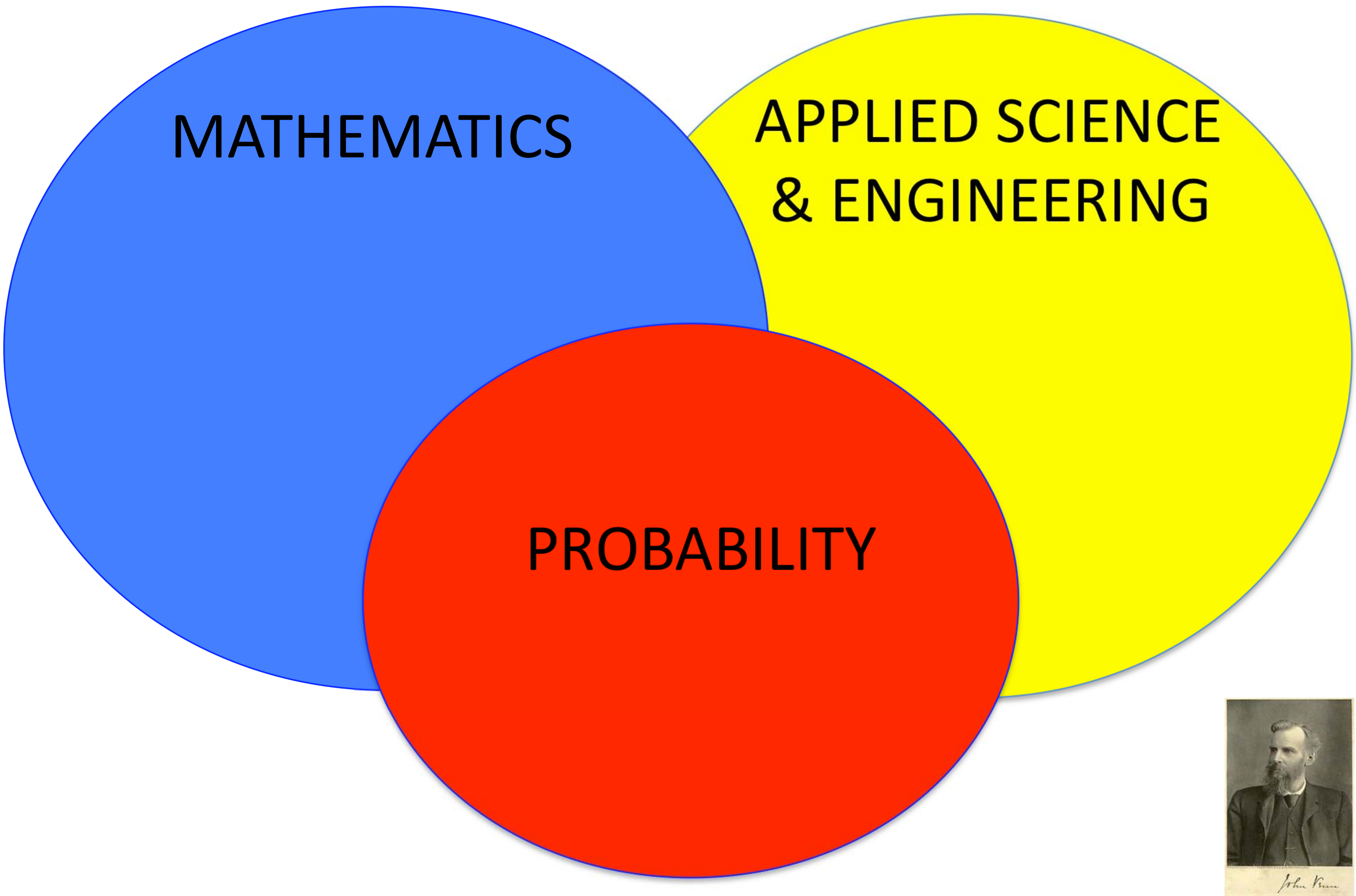
A RANDOM WALK THROUGH ANALYSIS, NETWORKS AND BIOLOGY



Ruth J Williams
G C Steward Visiting Fellow
Gonville and Caius College

April 2010

CONNECTIONS

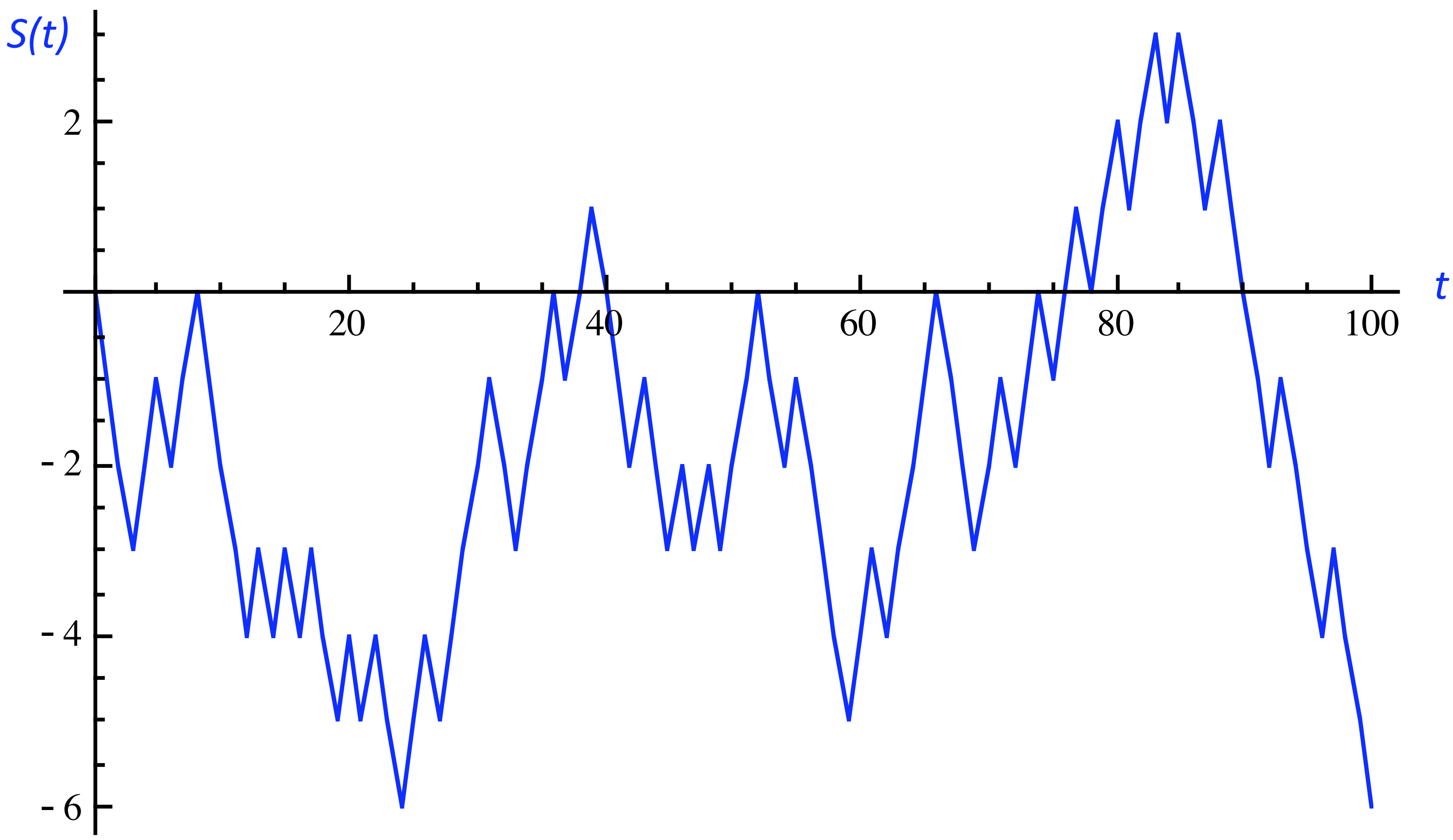


CONNECTIONS

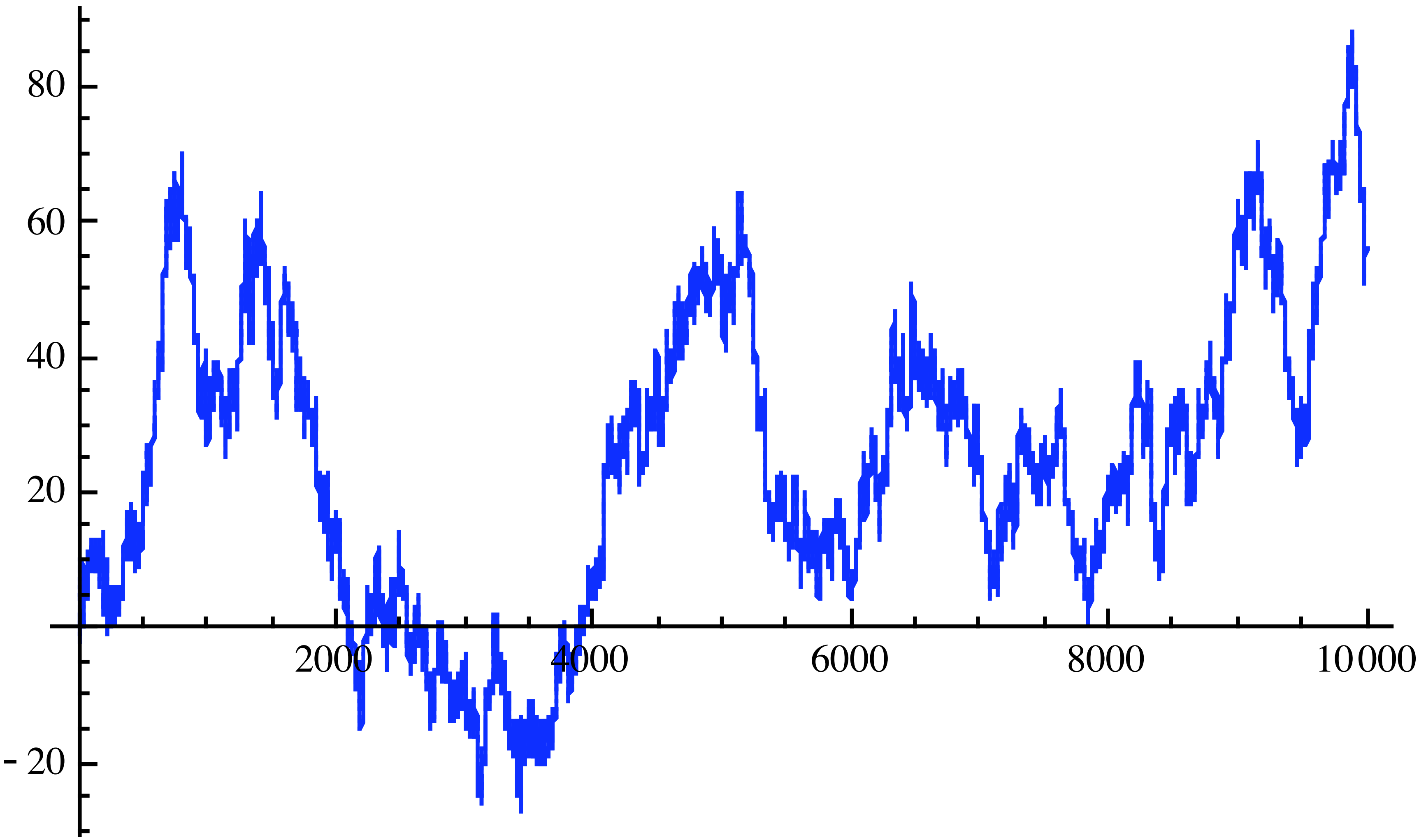
- Brownian motion and analysis
- Reflecting Brownian motion and queuing networks
- Queues and biology

BROWNIAN MOTION

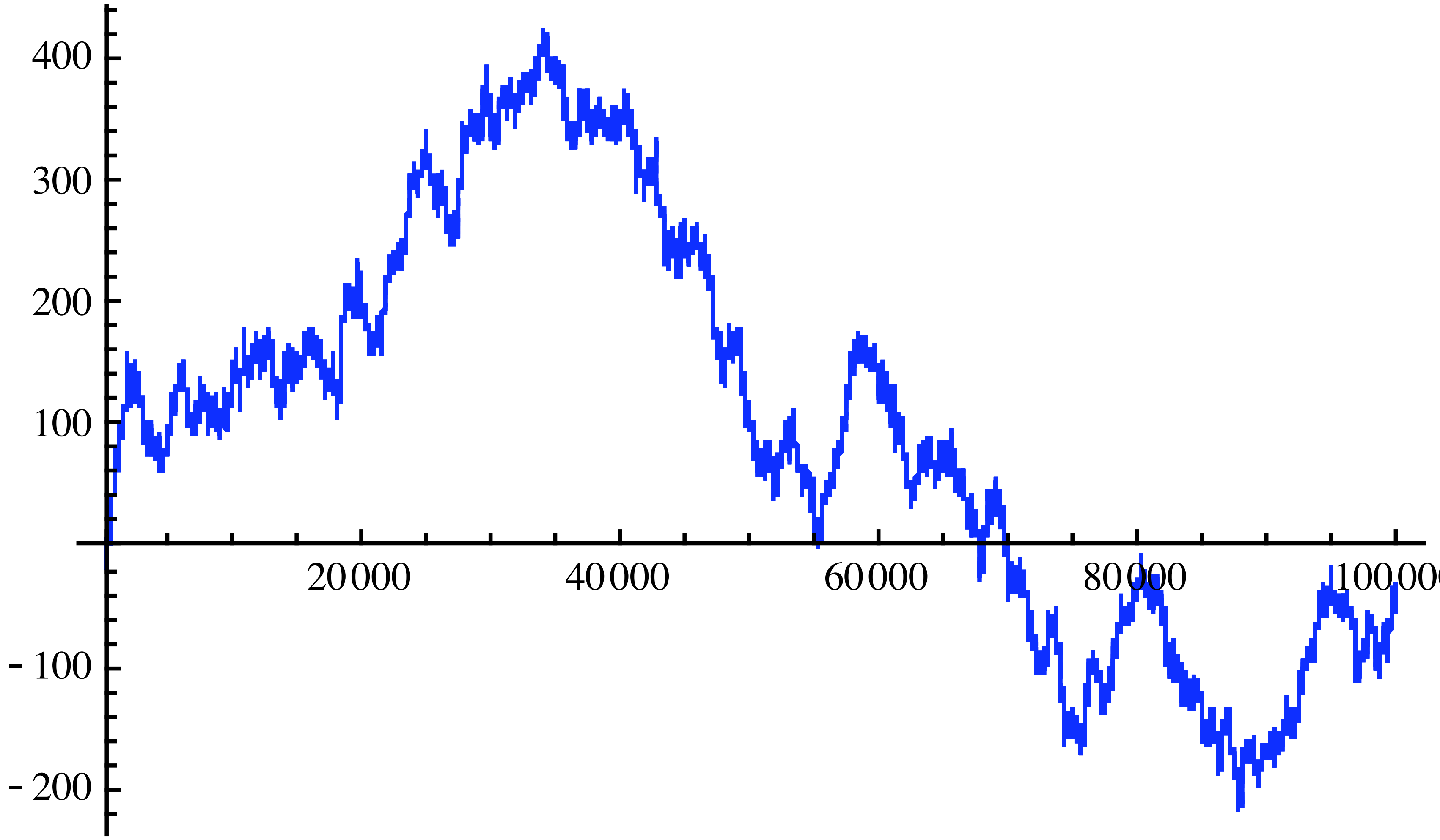
RANDOM WALK SIMULATIONS



RANDOM WALK SIMULATIONS



RANDOM WALK SIMULATIONS



RESCALED RANDOM WALK CONVERGES TO BROWNIAN MOTION

$$\hat{S}^m(\cdot) \triangleq \frac{S(m\cdot)}{\sqrt{m}} \Rightarrow B(\cdot) \text{ as } m \to \infty$$

VARIATION OF BROWNIAN PATHS

- Consider a partition π^n of [0,*t*]: $0 = t_0^n < t_1^n < \ldots < t_n^n = t$

$$\sum_{t_i^n, t_{i+1}^n \in \pi^n} \left| B(t_{i+1}^n) - B(t_i^n) \right| \to \infty \text{ as } \left| \pi^n \right| \to 0$$

VARIATION OF BROWNIAN PATHS

- Consider a partition π^n of [0,*t*]: $0 = t_0^n < t_1^n < \ldots < t_n^n = t$

$$\sum_{t_i^n, t_{i+1}^n \in \pi^n} \left| B(t_{i+1}^n) - B(t_i^n) \right| \to \infty \text{ as } \left| \pi^n \right| \to 0$$

$$\sum_{t_i^n, t_{i+1}^n \in \pi^n} (B(t_{i+1}^n) - B(t_i^n))^2 \to t \text{ as } \left| \pi^n \right| \to 0$$

VARIATION OF BROWNIAN PATHS

- Consider a partition π^n of [0,*t*]: $0 = t_0^n < t_1^n < \ldots < t_n^n = t$

$$\sum_{t_i^n, t_{i+1}^n \in \pi^n} \left| B(t_{i+1}^n) - B(t_i^n) \right| \to \infty \text{ as } \left| \pi^n \right| \to 0$$

$$\sum_{t_i^n, t_{i+1}^n \in \pi^n} (B(t_{i+1}^n) - B(t_i^n))^2 \to t \text{ as } \left| \pi^n \right| \to 0$$

- Brownian paths are not of finite variation but they are of finite quadratic variation

VARIATION OF BROWNIAN PATHS

- Consider a partition π^n of [0,*t*]: $0 = t_0^n < t_1^n < \ldots < t_n^n = t$

$$\sum_{t_i^n, t_{i+1}^n \in \pi^n} \left| B(t_{i+1}^n) - B(t_i^n) \right| \to \infty \text{ as } \left|\pi^n\right| \to 0$$

$$\sum_{t_i^n, t_{i+1}^n \in \pi^n} (B(t_{i+1}^n) - B(t_i^n))^2 \to t \text{ as } \left|\pi^n\right| \to 0$$

- Brownian paths are not of finite variation but they are of finite quadratic variation

Formally: $"(dB)^2 = dt"$

BROWNIAN MOTION

- Limit of renormalized random walks
- Scaling property: $B(a\cdot) \stackrel{d}{=} a^{1/2}B(\cdot),\ a > 0$
- Continuous sample paths, Markov process
- Used as the source of randomness in many models in applications (e.g., neuroscience, communications, finance)

ITO's STOCHASTIC CALCULUS

STOCHASTIC CALCULUS

Let $f:\mathbb{R}\to\mathbb{R}$ be twice continuously differentiable

$$f(B(t))-f(B(0))=\sum_{t_{i+1}^n,t_i^n\in\pi^n}(f(B(t_{i+1}^n)-f(B(t_i^n))$$

STOCHASTIC CALCULUS

Let $f:\mathbb{R}\to\mathbb{R}$ be twice continuously differentiable

$$f(B(t))-f(B(0))=\sum_{t_{i+1}^n,t_i^n\in\pi^n}(f(B(t_{i+1}^n)-f(B(t_i^n))$$

$$=\sum_{t_{i+1}^n,t_i^n\in\pi^n}\{f'(B(t_i^n))(B(t_{i+1}^n)-B(t_i^n))$$

$$+\tfrac{1}{2}f''(B(\tilde{t}_i^n))(B(t_{i+1}^n)-B(t_i^n))^2\}$$

STOCHASTIC CALCULUS

Let $f:\mathbb{R}\to\mathbb{R}$ be twice continuously differentiable

$$f(B(t))-f(B(0))=\sum_{t_{i+1}^n,t_i^n\in\pi^n}(f(B(t_{i+1}^n)-f(B(t_i^n))$$

$$=\sum_{t_{i+1}^n,t_i^n\in\pi^n}\{f'(B(t_i^n))(B(t_{i+1}^n)-B(t_i^n))$$

$$+\tfrac{1}{2}f''(B(\tilde{t}_i^n))(B(t_{i+1}^n)-B(t_i^n))^2\}$$

$$\to\int_0^t f'(B(s))\,dB(s)+\tfrac{1}{2}\int_0^t f''(B(s))\,ds \text{ as } |\pi^n|\to 0$$

ITO'S FORMULA

For $f:\mathbb{R}\to\mathbb{R}$ twice continuously differentiable

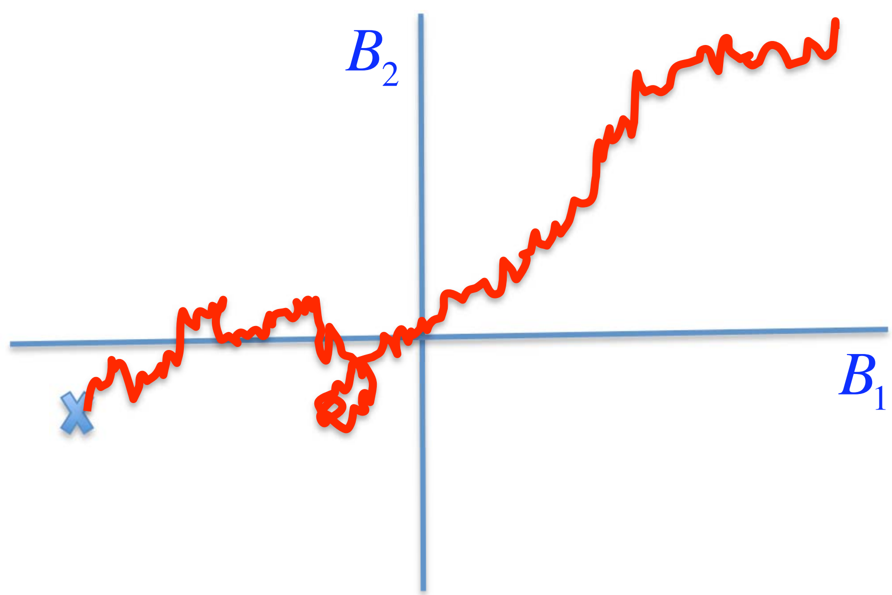
$$f(B(t))-f(B(0))=\int_0^t f'(B(s))\,dB(s)+\tfrac{1}{2}\int_0^t f''(B(s))\,ds$$

MULTIDIMENSIONAL BROWNIAN MOTION

- *d*-dimensional Brownian motion $B = (B_1, \ldots, B_d)$ where $B_1, \ldots, B_d$ are independent one dimensional Brownian motions

MULTIDIMENSIONAL BROWNIAN MOTION

- *d*-dimensional Brownian motion $B = (B_1, \ldots, B_d)$ where $B_1, \ldots, B_d$ are independent one dimensional Brownian motions



MULTIDIMENSIONAL ITO FORMULA

For $f \in C^2(\mathbb{R}^d)$,

$$f(B(t)) - f(B(0)) = \int_0^t \nabla f(B(s)) \cdot dB(s) + \tfrac{1}{2}\int_0^t \Delta f(B(s))\,ds$$

where $\nabla f = \begin{pmatrix} \dfrac{\partial f}{\partial x_1} \\ \vdots \\ \dfrac{\partial f}{\partial x_d} \end{pmatrix}$ and $\Delta f = \sum_{i=1}^{d} \dfrac{\partial^2 f}{\partial x_i^2}$

MULTIDIMENSIONAL ITO FORMULA

For $f \in C^2(\mathbb{R}^d)$,

$$f(B(t)) - f(B(0)) = \int_0^t \nabla f(B(s)) \cdot dB(s) + \tfrac{1}{2}\int_0^t \Delta f(B(s))\,ds$$

If ∇f is bounded, then for all $t \geq 0$,

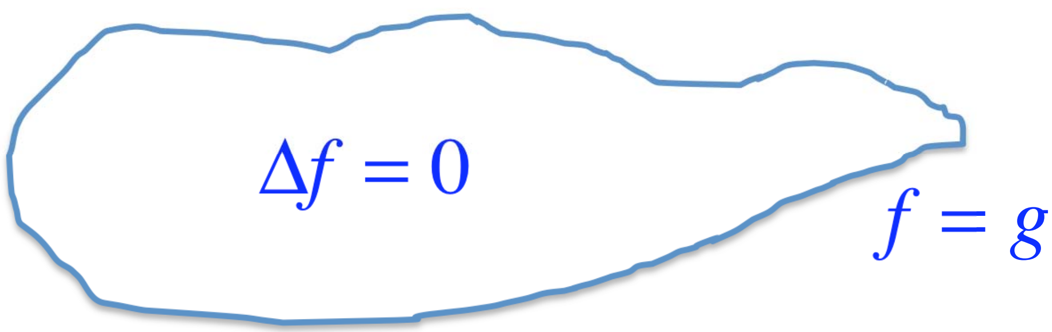
$$E_x\left[\int_0^t \nabla f(B(s)) \cdot dB(s)\right] = 0$$

A PARTIAL DIFFERENTIAL EQUATION CONNECTION

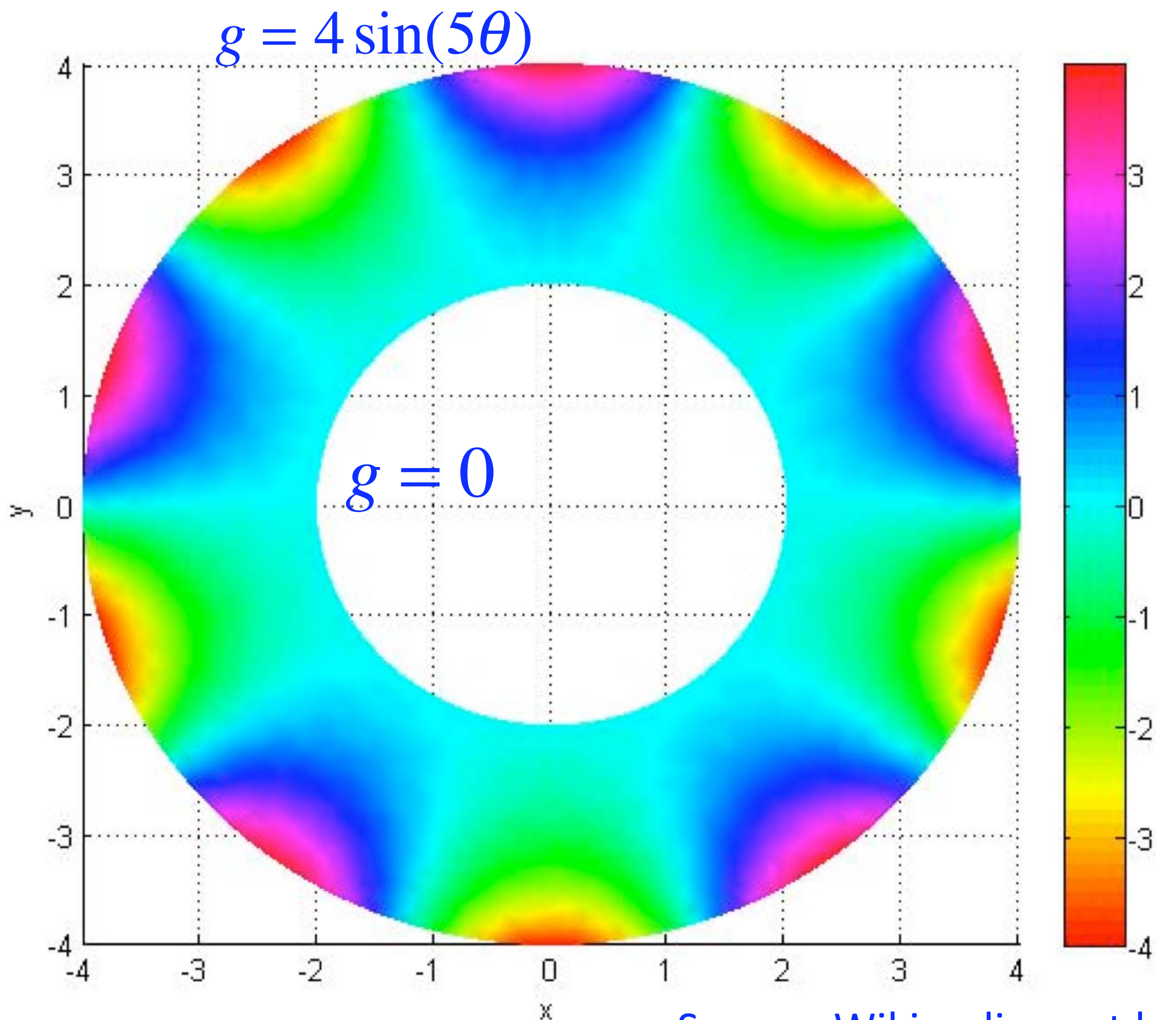
DIRICHLET PROBLEM

- Given a smooth bounded domain *D* in $\mathbb{R}^d$
- Given g a continuous function on the boundary ∂D
- Seek f continuous on $\bar{D}$ satisfying

$$\Delta f = 0 \quad \text{in} \quad D$$
$$f = g \quad \text{on} \; \partial D$$



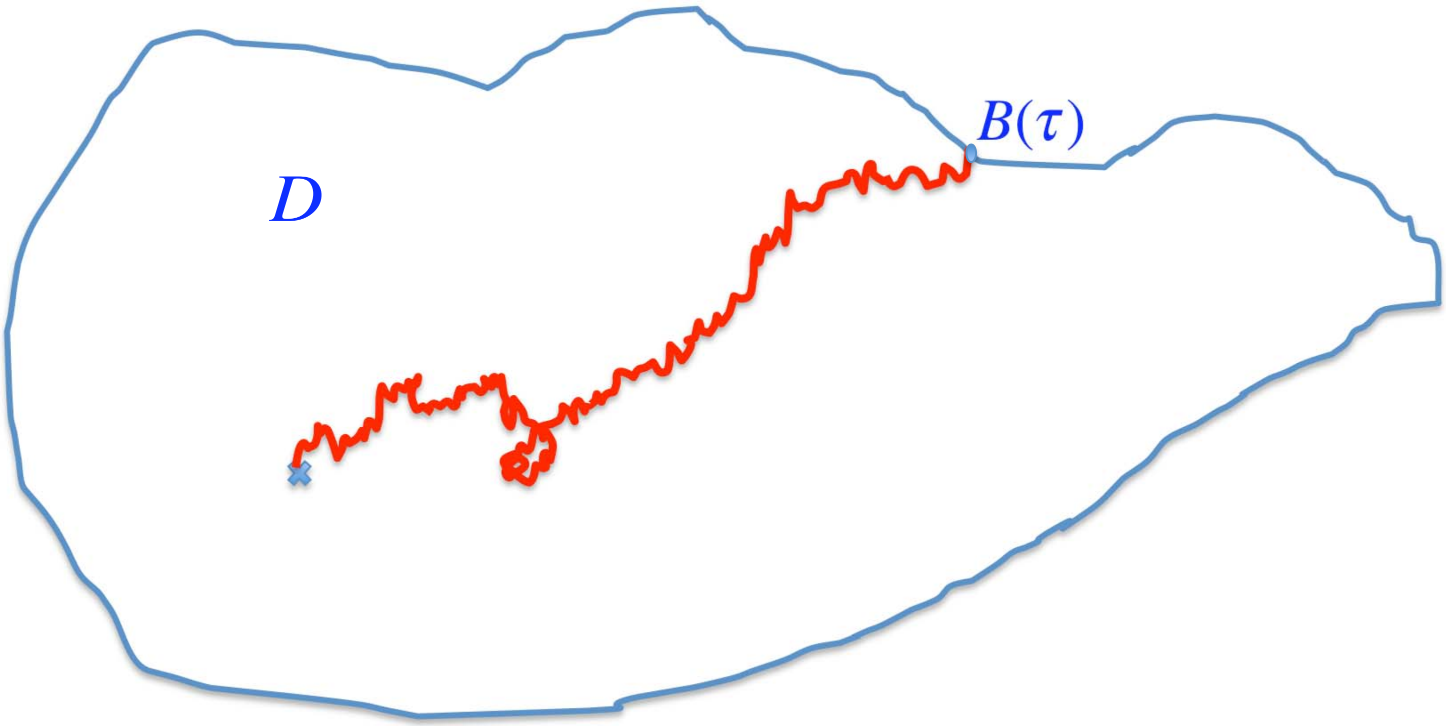
AN EXAMPLE OF SOLUTION OF THE DIRICHLET PROBLEM ON AN ANNULUS



Source: Wikipedia post by Davidian Skitzou

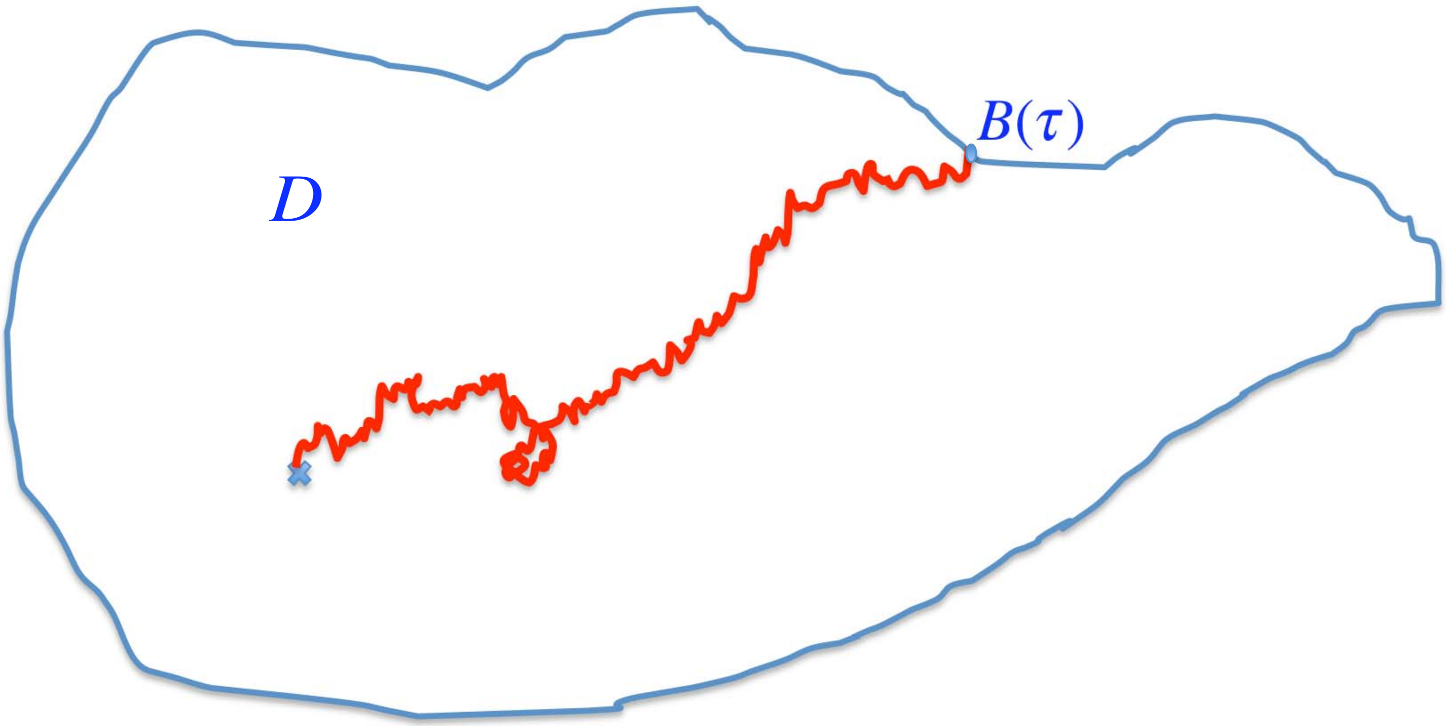
SOLUTION VIA BROWNIAN MOTION

$$\tau = \inf\{t > 0 : B(t) \notin D\}$$



SOLUTION VIA BROWNIAN MOTION

$$\tau = \inf\{t > 0 : B(t) \notin D\}$$



$$f(x) = E_x[g(B(\tau))]$$

SOLUTION

Theorem

A function f is a solution of the Dirichlet problem if and only if

$$f(x) = E_x[g(B(\tau))] \text{ for all } x \in \bar{D}$$

Idea of proof of probabilistic representation

- Let f be a (smooth) solution of the Dirichlet problem

Idea of proof of probabilistic representation

- Let f be a (smooth) solution of the Dirichlet problem

- By Ito's formula, for all t,

$$f(B(t)) = f(B(0)) + \int_0^t \nabla f(B(s)) \cdot dB(s) + \tfrac{1}{2} \int_0^t \Delta f(B(s)) ds$$

Idea of proof of probabilistic representation

- Let f be a (smooth) solution of the Dirichlet problem

- By Ito's formula, for all t,

$$f(B(t)) = f(B(0)) + \int_0^t \nabla f(B(s)) \cdot dB(s) + \tfrac{1}{2}\int_0^t \Delta f(B(s))ds$$

- So,

$$f(B(t \wedge \tau)) = f(B(0)) + \int_0^{t \wedge \tau} \nabla f(B(s)) \cdot dB(s) + \tfrac{1}{2}\int_0^{t \wedge \tau} \Delta f(B(s))ds$$

Idea of proof of probabilistic representation

- Let f be a (smooth) solution of the Dirichlet problem

- By Ito's formula, for all t,

$$f(B(t)) = f(B(0)) + \int_0^t \nabla f(B(s)) \cdot dB(s) + \tfrac{1}{2}\int_0^t \Delta f(B(s))ds$$

- So,

$$f(B(t\wedge\tau)) = f(B(0)) + \int_0^{t\wedge\tau} \nabla f(B(s)) \cdot dB(s) + \tfrac{1}{2}\int_0^{t\wedge\tau} \Delta f(B(s))ds$$

0

Idea of proof of probabilistic representation

- Let f be a (smooth) solution of the Dirichlet problem

- By Ito's formula, for all *t*,

$$f(B(t)) = f(B(0)) + \int_0^t \nabla f(B(s)) \cdot dB(s) + \tfrac{1}{2}\int_0^t \Delta f(B(s))ds$$

- So,

$$f(B(t\wedge\tau)) = f(B(0)) + \int_0^{t\wedge\tau} \nabla f(B(s)) \cdot dB(s) + \tfrac{1}{2}\int_0^{t\wedge\tau} \Delta f(B(s))ds$$

0

- Taking expectations:

$$E_x[f(B(t\wedge\tau))] = E_x[f(B(0))] = f(x)$$

Idea of proof of probabilistic representation

- Let f be a (smooth) solution of the Dirichlet problem

- By Ito's formula, for all t,

$$f(B(t)) = f(B(0)) + \int_0^t \nabla f(B(s)) \cdot dB(s) + \tfrac{1}{2}\int_0^t \Delta f(B(s))ds$$

- So,

$$f(B(t \wedge \tau)) = f(B(0)) + \int_0^{t \wedge \tau} \nabla f(B(s)) \cdot dB(s) + \tfrac{1}{2}\int_0^{t \wedge \tau} \Delta f(B(s))ds$$

0

- Taking expectations:

$$E_x[f(B(t \wedge \tau))] = E_x[f(B(0))] = f(x)$$

- Let $t \longrightarrow \infty$,

$$E_x[f(B(\tau))] = f(x)$$

CONNECTIONS

- Brownian motion and analysis

http://www.math.ucsd.edu/~williams/talks/caius/gcsteward2010.html

- Reflecting Brownian motion and queuing networks
- Queues and biology

THANK YOU