A RANDOM WALK THROUGH ANALYSIS, NETWORKS AND BIOLOGY

In this series of three lectures, I will describe some connections between probability and other fields. I will begin by introducing the fundamental stochastic process of Brownian motion and will illustrate some connections to partial differential equations via Ito's stochastic calculus. I will then introduce a variant called reflecting Brownian motion which arises in applications to queueing networks. Finally, I will illustrate a connection between the probability theory of queues and synthetic biology.
A RANDOM WALK THROUGH ANALYSIS, NETWORKS AND BIOLOGY

Ruth J Williams
G C Steward Visiting Fellow
Gonville and Caius College
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CONNECTIONS

MATHEMATICS

APPLIED SCIENCE & ENGINEERING

PROBABILITY
CONNECTIONS

• Brownian motion and analysis
• Reflecting Brownian motion and queuing networks
• Queues and biology
BROWNIAN MOTION
RANDOM WALK SIMULATIONS
RESCALED RANDOM WALK CONVERGES TO BROWNIAN MOTION

\[ \hat{S}^{m}(\cdot) \triangleq \frac{S(m\cdot)}{\sqrt{m}} \Rightarrow B(\cdot) \text{ as } m \to \infty \]
VARIATION OF BROWNIAN PATHS

- Consider a partition $\pi^n$ of $[0,t]$: $0 = t_0^n < t_1^n < \ldots < t_n^n = t$

\[\sum_{t_i^n, t_{i+1}^n \in \pi^n} \left| B(t_{i+1}^n) - B(t_i^n) \right| \to \infty \text{ as } |\pi^n| \to 0\]
VARIATION OF BROWNIAN PATHS

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$$\sum_{t_i^n, t_{i+1}^n \in \pi^n} |B(t_{i+1}^n) - B(t_i^n)| \rightarrow \infty \text{ as } |\pi^n| \rightarrow 0$$

$$\sum_{t_i^n, t_{i+1}^n \in \pi^n} (B(t_{i+1}^n) - B(t_i^n))^2 \rightarrow t \text{ as } |\pi^n| \rightarrow 0$$
VARIATION OF BROWNIAN PATHS

• Consider a partition $\pi^n$ of $[0,t]$: $0 = t^n_0 < t^n_1 < ... < t^n_n = t$

$$\sum_{t^n_i, t^n_{i+1} \in \pi^n} \left| B(t^n_{i+1}) - B(t^n_i) \right| \to \infty \text{ as } |\pi^n| \to 0$$

$$\sum_{t^n_i, t^n_{i+1} \in \pi^n} (B(t^n_{i+1}) - B(t^n_i))^2 \to t \text{ as } |\pi^n| \to 0$$

• Brownian paths are not of finite variation but they are of finite quadratic variation
VARIATION OF BROWNIAN PATHS

• Consider a partition $\pi^n$ of $[0,t]$: $0 = t_0^n < t_1^n < ... < t_n^n = t$

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• Brownian paths are not of finite variation but they are of finite quadratic variation

Formally: "$(dB)^2 = dt$"
BROWNIAN MOTION

• Limit of renormalized random walks
• Scaling property: $B(a\cdot) = a^{1/2} B(\cdot), \ a > 0$
• Continuous sample paths, Markov process
• Used as the source of randomness in many models in applications (e.g., neuroscience, communications, finance)
ITO’s STOCHASTIC CALCULUS
STOCHASTIC CALCULUS

Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be twice continuously differentiable

\[
f(B(t)) - f(B(0)) = \sum_{t_{i+1}^n, t_i^n \in \pi^n} (f(B(t_{i+1}^n)) - f(B(t_i^n))
\]
Let \( f : \mathbb{R} \to \mathbb{R} \) be twice continuously differentiable

\[
f(B(t)) - f(B(0)) = \sum_{t_i^n \in \pi^n} (f(B(t_{i+1}^n)) - f(B(t_i^n)))
\]

\[
= \sum_{t_i^n, t_{i+1}^n \in \pi^n} \left\{ f'(B(t_i^n))(B(t_{i+1}^n) - B(t_i^n)) + \frac{1}{2} f''(B(\tilde{t}_i^n))(B(t_{i+1}^n) - B(t_i^n))^2 \right\}
\]
STOCHASTIC CALCULUS

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be twice continuously differentiable

$$f(B(t)) - f(B(0)) = \sum_{t_{i+1}^n, t_i^n \in \pi^n} (f(B(t_{i+1}^n)) - f(B(t_i^n)))$$

$$= \sum_{t_{i+1}^n, t_i^n \in \pi^n} \{f'(B(t_i^n))(B(t_{i+1}^n) - B(t_i^n))$$

$$+ \frac{1}{2} f''(B(t_i^n))(B(t_{i+1}^n) - B(t_i^n))^2 \}$$

$$\rightarrow \int_0^t f'(B(s)) dB(s) + \frac{1}{2} \int_0^t f''(B(s)) ds \quad \text{as} \quad |\pi^n| \rightarrow 0$$
For $f: \mathbb{R} \to \mathbb{R}$ twice continuously differentiable

$$f(B(t)) - f(B(0)) = \int_0^t f'(B(s)) dB(s) + \frac{1}{2} \int_0^t f''(B(s)) ds$$
MULTIDIMENSIONAL BROWNIAN MOTION

• $d$-dimensional Brownian motion $B = (B_1, ..., B_d)$ where $B_1, ..., B_d$ are independent one dimensional Brownian motions
MULTIDIMENSIONAL BROWNIAN MOTION

• \(d\)-dimensional Brownian motion \(B = (B_1, \ldots, B_d)\)
  where \(B_1, \ldots, B_d\) are independent one dimensional Brownian motions
MULTIDIMENSIONAL ITO FORMULA

For $f \in C^2(\mathbb{R}^d)$,

$$f(B(t)) - f(B(0)) = \int_0^t \nabla f(B(s)) \cdot dB(s) + \frac{1}{2} \int_0^t \Delta f(B(s)) ds$$

where $\nabla f = \left( \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_d} \right)$ and $\Delta f = \sum_{i=1}^d \frac{\partial^2 f}{\partial x_i^2}$.
MULTIDIMENSIONAL ITO FORMULA

For $f \in C^2(\mathbb{R}^d)$,

$$f(B(t)) - f(B(0)) = \int_0^t \nabla f(B(s)) \cdot dB(s) + \frac{1}{2} \int_0^t \Delta f(B(s)) ds$$

If $\nabla f$ is bounded, then for all $t \geq 0$,

$$E_x \left[ \int_0^t \nabla f(B(s)) \cdot dB(s) \right] = 0$$
A PARTIAL DIFFERENTIAL EQUATION CONNECTION
DIRICHLET PROBLEM

• Given a smooth bounded domain $D$ in $\mathbb{R}^d$
• Given $g$ a continuous function on the boundary $\partial D$
• Seek $f$ continuous on $\overline{D}$ satisfying

\[ \Delta f = 0 \quad \text{in} \quad D \]
\[ f = g \quad \text{on} \quad \partial D \]
AN EXAMPLE OF SOLUTION OF THE DIRICHLET PROBLEM ON AN ANNULUS

\[ g = 4 \sin(5\theta) \]

\[ g = 0 \]

Source: Wikipedia post by Davidian Skitzou
SOLUTION VIA BROWNIAN MOTION

$$\tau = \inf\{t > 0 : B(t) \notin D\}$$
SOLUTION VIA BROWNIAN MOTION

\[ \tau = \inf \{ t > 0 : B(t) \notin D \} \]

\[ f(x) = E_x [ g(B(\tau)) ] \]
SOLUTION

Theorem

A function $f$ is a solution of the Dirichlet problem if and only if

$$f(x) = E_x[g(B(\tau))] \text{ for all } x \in \bar{D}$$
Idea of proof of probabilistic representation

• Let $f$ be a (smooth) solution of the Dirichlet problem
Idea of proof of probabilistic representation

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• By Ito’s formula, for all $t$,

$$f(B(t)) = f(B(0)) + \int_0^t \nabla f(B(s)) \cdot dB(s) + \frac{1}{2} \int_0^t \Delta f(B(s)) ds$$
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• So,

$$f(B(t \wedge \tau)) = f(B(0)) + \int_0^{t \wedge \tau} \nabla f(B(s)) \cdot dB(s) + \frac{1}{2} \int_0^{t \wedge \tau} \Delta f(B(s)) ds$$
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• Taking expectations:

$$E_x[f(B(t \wedge \tau))] = E_x[f(B(0))] = f(x)$$
Idea of proof of probabilistic representation

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  \]

- Taking expectations:
  \[
  E_x[f(B(t \wedge \tau))] = E_x[f(B(0))] = f(x)
  \]

- Let $t \to \infty$, 
  \[
  E_x[f(B(\tau))] = f(x)
  \]
CONNECTIONS

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- Reflecting Brownian motion and queuing networks

- Queues and biology
THANK YOU