Controlled and constrained martingale problems

Controlled martingle problems

Let E and U be compact and

$$A \subset C_b(E) \times C_b(E \times U)$$

The martingale problem: Find a control process u and a state process X for which there exists a filtration $\{\mathcal{F}_t\}$ such that u and X are $\{\mathcal{F}_t\}$-adapted and for each $f \in \mathcal{D}(A)$

$$M_f(t) = f(X(t)) - f(X(0)) - \int_0^t Af(X(s), u(s))ds \qquad (1)$$

is a $\{\mathcal{F}_t\}$-martingale. Let $\mathcal{S}_A$ be the collection of all such solutions.

Cost minimization: For example: Find a solution that achieves

$$\min_{(X,U)} \mathbb{E}[\int_0^\infty e^{-\alpha s} c(X(s), u(s))ds] \qquad (2)$$

In what, if any, sense is the collection of solutions compact?

Relative compactness

Let X_n be a solution of the martingale problem for A_n, in particular, for $f \in \mathcal{D}(A_n)$,

$$M_f^n(t) = f(X_n(t)) - f(X_n(0)) - \int_0^t A_n f(X_n(s))ds$$

is a martingale.

Theorem 1 *Suppose E is compact (for example, $E = \mathbb{R}^d \cup \{\infty\}$) and for each f in some dense subset $\mathcal{D} \subset C(E)$ there exist $f_n \in \mathcal{D}(A_n)$ such that $f_n \to f$ uniformly and $\sup_n \|A_n f_n\| < \infty$, then $\{X_n\}$ is relatively compact in $D_E[0, \infty)$.*

Proof. The result follows from Theorems 3.9.4 and 3.9.1 of Ethier and Kurtz (1986). □

Measure determined by control

$$\Lambda([0,t] \times C) = \int_0^t \mathbf{1}_C(u(s))ds, \quad t \geq 0, C \in \mathcal{B}(U). \tag{3}$$

A bounded collections of measures on a compact set K is relatively compact in the weak topology, that is, there is a sequence of measures μ_n in the bounded collection such that there exists $\mu \in \mathcal{M}_f(K)$ such that

$$\int_K f(z)\mu_n(dz) \to \int_K f(z)\mu(dz).$$

So for a sequence $\{\Lambda_n\}$ of the form (3), there exists a measure Λ and a subsequence of $\{\Lambda_n\}$ such that for each $f \in C([0,\infty) \times U)$ with $f(t,u) = 0$ for $t > t_f$,

$$\int_{[0,\infty) \times U} f(s,u)\Lambda_n(ds \times du) \to \int_{[0,\infty) \times U} f(s,u)\Lambda(ds \times du)$$

Relaxed controls

Let $\mathcal{L}_m(U)$ be the collection of measures on $[0, \infty) \times U$ satisfying $\lambda([0, t] \times U) = t$, then $\mathcal{L}_m(U)$ (with the topology described above) is compact.

The collection $\Pi_0 \subset \mathcal{P}(D_E[0, \infty) \times \mathcal{L}_m(U))$ of distributions of solutions of the controlled martingale problem of the form (1) is relatively compact and every limit point has the property that there exists a filtration $\{\mathcal{F}_t\}$ such that for each $f \in \mathcal{D}(A)$

$$
\begin{aligned}
M_f(t) &= f(X(t)) - f(X(0)) - \int_0^t Af(X(s), u)\Lambda(ds \times du) \\
&= f(X(t)) - f(X(0)) - \int_0^t Af(X(s), u(s))\Lambda_s(du)ds
\end{aligned}
$$

is a $\{\mathcal{F}_t\}$-martingale. Let Π be the closure of Π_0.

Controlled forward equations

For $P \in \Pi$, let $\mu_t(dx \times du)$ be the measure determined by

$$\int_{E \times U} f(x, u)\mu_t(dx \times du) = \mathbb{E}^P[\int_U f(X(t), u)\Lambda_t(du)],$$

and let ν_t be the E-marginal. Then $\{\mu_t\}$ satisfies the controlled forward equation

$$\nu_t f = \nu_0 f + \int_0^t \mu_s Af ds$$

and under essentially the same technical conditions as in the uncontrolled case, in particular, $\mathcal{D}(A)$ closed under multiplication and separates points and $(1, 0) \in A$, every solution of the forward equation corresponds to a solution of the controlled martingale problem.

Kurtz and Stockbridge (1998, 1999, 2001)

A linear programming problem

Assuming existence for the controlled martingale problem (say with piecewise constant controls), the optimal solution corresponds to the solution of a linear programing problem: Find $\mu : [0, \infty) \to \mathcal{P}(E \times U)$ that minimizes

$$\int_0^\infty e^{-\alpha t} \mu_t c \, dt$$

subject to the requirements that

$$\nu_t f = \nu_0 f + \int_0^t \mu_s A f \, ds,$$

where ν_t is the E-marginal of μ_t.

Manne (1960); Kurtz and Stockbridge (1998, 1999, 2001)

Equivalent form

$$\inf_{\widehat{\pi} \in \mathcal{P}(E \times U)} \frac{1}{\alpha} \int_{E \times U} c(x, u) \widehat{\pi}(dx \times du)$$

subject to

$$\int_{E \times U} \left(Af(x, u) + \alpha \left[\int_{E} f(y) \nu_0(dy) - f(x) \right] \right) \widehat{\pi}(dx \times du) = 0, \quad f \in \mathcal{D}(A).$$

Constrained martingale problems

E compact (think $E = \mathbb{R}^d \cup \{\infty\}$), $E_0 \subset E$, open, A, the generator for a Markov process on E. For example,

$$Af(x) = \frac{1}{2} \sum_{i,j} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} f(x) + \sum_i b_i(x) \frac{\partial}{\partial x_i} f(x), \quad \mathcal{D}(A) = C_c^2(\mathbb{R}^d).$$

A determines the behavior of the process in E_0.

B, the generator of a Markov process (almost) which determines the behavior of the process in E_0^c and "constrains" the process to stay in $\overline{E}_0$. For example,

$$Bf(x) = \gamma(x) \cdot \nabla f(x),$$

where γ determines the direction a constraining "force" pushes when the process is on ∂E_0.

A controlled martingale problem

Let $Cf(y, u, v) = vAf(y) + (1-v)Bf(y, u)$ with controls $(u, v) \in U \times [0, 1]$. We allow relaxed controls so the formulation of the martingale problem becomes

Definition 2 (Y, V, μ), *with* $Y \in D_E[0, \infty)$, *and* μ *a* $\mathcal{P}(U)$-*valued process, is a solution of the controlled martingale problem if there exists a filtration* $\{\mathcal{F}_t\}$ *such that* (Y, V, μ) *is* $\{\mathcal{F}_t\}$-*adapted and*

$$f(Y(t)) - f(Y(0)) - \int_0^t V(s)Af(Y(s))ds - \int_0^t \int_U (1-V(s))Bf(Y(s), u)\mu_s(du)ds$$

is an $\{\mathcal{F}_t\}$-*martingale for all* $f \in \mathcal{D} \equiv \mathcal{D}(A) \cap \mathcal{D}(B)$.

The choice of controls must be restricted so that $V(t) = 1$ if $Y(t) \in E_0$, $V(t) = 0$ if $Y(t) \in \overline{E}_0^c$, $0 \le V(t) \le 1$ if $Y(t) \in \partial E_0$, and

$$\int_U \mathbf{1}_{\Xi(Y(s))}(u)\mu_s(du) = 1.$$

Reflecting diffusions

Suppose

$$Af(x) = \sum_{i,j} \frac{1}{2} a_{ij}(x) \partial_i \partial_j f(x) + \sum_i b_i(x) \partial_i f(x)$$

and $Bf(x, u) = u \cdot \nabla f(x)$. Let

$$\lambda_0(t) = \int_0^t V(s) ds \qquad \lambda_1(t) = \int_0^t (1 - V(s)) ds$$

Then the martingale is

$$f(Y(t)) - f(Y(0)) - \int_0^t Af(Y(s)) d\lambda_0(s) - \int_0^t \int_U u\mu_s(du) \cdot \nabla f(Y(s)) d\lambda_1(s).$$

Of course

$$\lambda_0(t) + \lambda_1(t) = t.$$

Time change

Assuming $\Xi(y) = \{\kappa(y)\}$, we have martingales

$$f(Y(t)) - f(Y(0)) - \int_0^t Af(Y(s))d\lambda_0(s) - \int_0^t \kappa(Y(s)) \cdot \nabla f(Y(s))d\lambda_1(s),$$

where $\lambda_0(t) + \lambda_1(t) = t$. If the boundary is smooth with $n(y)$ the inward normal at $y \in \partial E_0$, and $\kappa(y) \cdot n(y) > 0$, $y \in \partial E_0$, then λ_0 is strictly increasing and $\tau(t) = \inf\{s : \lambda_0(s) > t\}$ is continuous. Define $X(t) = Y(\tau(t))$ and $\lambda(t) = \lambda_1(\tau(t))$. Then

$$f(X(t)) - f(X(0)) - \int_0^t Af(X(s))ds - \int_0^t \kappa(X(s)) \cdot \nabla f(X(s))d\lambda(s)$$

is a $\{\mathcal{F}_{\tau(t)}\}$-martingale (or perhaps a local martingale).

Controlled stochastic differential equation

The corresponding SDE should be

$$Y(t) = Y(0) + \int_0^t \sqrt{V(s)} \sum_j \sigma_j(Y(s)) dW_j(s) + \int_0^t V(s) b(Y(s)) ds$$

$$+ \int_0^t (1 - V(s)) \kappa(Y(s)) ds,$$

and by the same arguments used in Stroock and Varadhan (1979) or those in Kurtz (2011), every solution of the controlled martingale problem corresponds to a solution of the SDE.

Stochastic differential equation

Inverting λ_0 as above, $\tau(t) = \inf\{s : \lambda_0(s) > t\}$,

$$X(t) = X(0) + \int_0^t \sum_j \sigma_j(X(s))dW_j^V(s) + \int_0^t b(X(s))ds$$
$$+ \int_0^t \kappa(X(s))d\lambda(s),$$

where the

$$W_j^V(s) = \int_0^{\tau(t)} \sqrt{V(s)}dW_j(s)$$

are independent standard Brownian motions.

Nonlocal boundary conditions

As before

$$Af(x) = \frac{1}{2} \sum_{i,j} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} f(x) + \sum_i b_i(x) \frac{\partial}{\partial x_i} f(x), \quad \mathcal{D}(A) = C_c^2(\mathbb{R}^d),$$

but now take

$$Bf(x) = \int_{E_0} (f(y) - f(x)) \mu(x, dy)$$

where we assume $\mu(x, E_0) = 1$. The controlled martingale then becomes

$$f(Y(t)) - f(Y(0)) - \int_0^t V(s) Af(Y(s)) ds - \int_0^t (1 - V(s)) Bf(Y(s)) ds.$$

or

$$f(Y(t)) - f(Y(0)) - \int_0^t Af(Y(s)) d\lambda_0(s) - \int_0^t Bf(Y(s)) d\lambda_1(s).$$

Corresponding SDE

The controlled process will satisfy

$$Y(t) = Y(0) + \int_0^t \sqrt{V(s)} \sum_j \sigma_j(Y(s)) dW_j(s) + \int_0^t V(s) b(Y(s)) ds$$

$$+ \int_{[0,t] \times [0,\infty) \times [0,1]} \mathbf{1}_{[0,1-V(s)]}(u)(H(Y(s-), v) - Y(s-)) \xi(ds, du, dv)$$

As before, every solution of the controlled martingale problem corresponds to a solution of the controlled SDE, but λ_0 need not be (probably isn't) strictly increasing.

Stochastic equation for X

If, for the diffusion corresponding to A, $P\{\inf\{t : Z(t) \in \partial E_0\} = \inf\{t : Z(t) \in \overline{E}_0^c\} = 1$, then $Y(s) \in \partial E_0$ implies $V(s) = 0$ and $X(t) = Y(\tau(t))$ satisfies

$$X(t) = X(0) + \int_0^t \sum_j \sigma_j(X(s))dW_j^V(s) + \int_0^t b(X(s))ds$$
$$+ \int_0^t (H(X(s-), \xi_{N(s)}) - X(s-))dN(s)$$

where $N(t)$ counts the number times X hits ∂E_0 by time t, ξ_k is the v-coordinate at the kth jump time of Y (note that the ξ_k are independent, uniform $[0, 1]$) and, as before,

$$W_j^V(s) = \int_0^{\tau(t)} \sqrt{V(s)}dW_j(s)$$

are independent standard Brownian motions.

Markov selection

Under the compactness assumption on E, and assuming $\mathcal{D}$ is dense in $C(E)$, the set of solutions of the controlled martingale problem will be compact, and solutions will exist for all $\nu \in \mathcal{P}(E_2)$ for some $E_2 \supset \overline{E}_0$. Specifically, let $\Pi_\nu \subset \mathcal{P}(D_E[0,\infty) \times C_{[0,\infty)}[0,\infty))$ be the collection of distributions for (Y, λ_0) where $Y(0)$ has distribution ν.

Let $\{h_k\} \subset C(\overline{E}_0)$ and define $\Pi_\nu^{h_1} \subset \Pi_\nu$ to be the set of $P \in \Pi_\nu$ such that

$$E^P\Big[\int_0^\infty e^{-\lambda_0(s)} h_1(Y(s)) d\lambda_0(s)\Big] = \sup_{Q \in \Pi_\nu} E^Q\Big[\int_0^\infty e^{-\lambda_0(s)} h_1(Y(s)) d\lambda_0(s)\Big],$$

and recursively define $\Pi_\nu^{h_1,\dots,h_n}$ to be the set of $P \in \Pi_\nu^{h_1,\dots,h_{n-1}}$ such that

$$E^P\Big[\int_0^\infty e^{-\lambda_0(s)} h_n(Y(s)) d\lambda_0(s)\Big] = \sup_{Q \in \Pi_\nu^{h_1,\dots,h_{n-1}}} E^Q\Big[\int_0^\infty e^{-\lambda_0(s)} h_1(Y(s)) d\lambda_0(s)\Big].$$

Define $\Pi_\nu^\infty = \cap_n \Pi_\nu^{h_1,\dots,h_n}$.

Martingale properties

Let Γ^∞ be the collection of distributions $X(t) = Y(\tau(t))$, $\tau(t) = \inf\{s : \lambda_0(s) > t\}$ for (Y, λ_0) with distribution in $\Pi^\infty \equiv \cup_{\nu \in \mathcal{P}(E_2)} \Pi_\nu^\infty$. For $P \in \Pi_{\delta_x}^\infty$, define

$$u_{h_n}(x) \equiv E^P\Big[\int_0^\infty e^{-\lambda_0(s)} h_n(Y(s)) d\lambda_0(s)\Big] = E^P\Big[\int_0^\infty e^{-t} h_n(X(s)) ds\Big].$$

Then for every X obtained from a $P \in \Pi^\infty$,

$$u_{h_n}(X(s)) - \int_0^t (u_{h_n}(X(s)) - h_n(X(s))) ds$$

is a $\{\mathcal{F}_t^X\}$-martingale.

A generator and a Markov process

Assuming the linear span of $\{h_n\}$ is bp dense in $B(\overline{E}_0)$, for $h \in B(\overline{E}_0)$, define

$$u_h(x) = E^P\big[\int_0^\infty e^{-\lambda_0(s)} h(Y(s)) d\lambda_0(s)\big], \quad P \in \Pi_{\delta_x}^\infty.$$

Then

$$u_h(X(s)) - \int_0^t (u_h(X(s)) - h(X(s))) ds$$

is a $\{\mathcal{F}_t^X\}$-martingale, and uniqueness holds for the martingale problem for

$$\mathbb{A} = \{(u_h, u_h - h) : h \in B(\overline{E}_0)\}$$

which ensures that all such X are Markov.

Viscosity solutions for the H-Y range condition
Costantini and Kurtz (2015)

For each solution Y of the controlled martingale problem

$$f(Y(t)) - f(Y(0)) - \int_0^t Af(Y(s)) d\lambda_0(s) - \int_0^t \int_{\Xi(Y(s))} Bf(Y(s), u)\mu_s(du) d\lambda_1(s),$$

assuming $\lambda_0(s) \to \infty$, let $\tau(t) = \inf\{s : \lambda_0(s) > t\}$, and define

$$X(t) = Y(\tau(t)).$$

Then $\int_0^\infty e^{-\lambda_0(s)} h(Y(s)) d\lambda_0(s) = \int_0^\infty e^{-t} h(X(t)) dt$ and for $Y(0) = x$, we should have

$$E[\int_0^\infty e^{-\lambda_0(s)} h(Y(s)) d\lambda_0(s)] = E[\int_0^\infty e^{-t} h(X(t)) dt] = (I - \widehat{A})^{-1} h(x)$$

where $\widehat{A}$ is the generator for X,

$$\widehat{A} \supset \{(f, Af) : Bf = 0\}$$

Sub and super solutions

Let Π_x be the collection of distributions of solutions of the controlled martingale problem (Y, λ_0). Then assuming that for all $x \in \overline{E}_0$, Π_x is nonempty and compact and . . .

$$u_h^+(x) \equiv \sup_{P \in \Pi_x} E[\int_0^\infty e^{-\lambda_0(s)} h(Y(s)) d\lambda_0(s)]$$

is a subsolution of $(I - \widehat{A})u = h$ in the sense that u_h^+ is upper semi-continuous and if $f \in \mathcal{D}$ and $x_0 \in \overline{E}_0$ satisfy

$$\sup_x (u^+ - f)(x) = (u^+ - f)(x_0), \tag{4}$$

then

$$\lambda u^+(x_0) - Af(x_0) \leq h(x_0), \qquad \text{if } x_0 \in E_0,$$
$$(\lambda u^+(x_0) - Af(x_0) - h(x_0)) \wedge (- \sup_{u \in \Xi(x_0)} Bf(x_0, u)) \leq 0, \qquad \text{if } x_0 \in \partial E_0.$$

Wentzell boundary conditions

Let B be the generator of a diffusion process Z such that $Z(0) \in \overline{E}_0$ implies $Z(t) \in E_0$, $t \geq 0$. The controlled martingale problem becomes

$$f(X(t)) - f(X(0)) - \int_0^t Af(X(s))d\lambda_0(s) - \int_0^t Bf(X(s))d\lambda_1(s)$$

where for $\mathcal{D} \equiv \mathcal{D}(A) = \mathcal{D}(B) = C_c^2(\mathbb{R}^d)$,

$$Af(x) = \frac{1}{2} \sum_{i,j} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} f(x) + \sum_i b_i(x) \frac{\partial}{\partial x_i} f(x)$$

and

$$Bf(x) = \frac{1}{2} \sum_{i,j} \alpha_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} f(x) + \sum_i \beta_i(x) \frac{\partial}{\partial x_i} f(x).$$

Existence

Let $Y^\epsilon(0)$ have distribution $\mu \in \mathcal{P}(\overline{E}_0)$ and evolve as a solution of the martingale problem for A until the first time τ_1^ϵ that Y^ϵ hits ∂E_0.

After time τ_1^ϵ, let Y^ϵ evolves as a solution of the martingale problem for B until $\sigma_1^\epsilon = \inf\{t > \tau_1^\epsilon : \inf_{x \in \partial E_0} |Y^\epsilon(t) - x| \geq \epsilon\}$.

By pasting, Y^ϵ is constructed so that for $f \in \mathcal{D}$,

$$f(Y^\epsilon(t)) - f(Y(0)) - \int_0^t \left(\sum_{k=0}^\infty \mathbf{1}_{[\sigma_k^\epsilon, \tau_{k+1}^\epsilon)}(s) A f(Y^\epsilon(s)) + \sum_{k=1}^\infty \mathbf{1}_{[\tau_k^\epsilon, \sigma_k^\epsilon)}(s) B f(Y^\epsilon(s)) \right) ds$$

is a martingale. Define $\lambda_0^\epsilon(t) = \int_0^t \sum_{k=0}^\infty \mathbf{1}_{[\sigma_k^\epsilon, \tau_{k+1}^\epsilon)}(s) ds$.

Then, $\{(Y^\epsilon, \lambda_0^\epsilon, \lambda_1^\epsilon), \epsilon > 0\}$ is relatively compact, and every limit point $(Y, \lambda_0, \lambda_1)$ will give a solution of the controlled martingale problem.

Corresponding SDE

The controlled process will satisfy

$$Y(t) = Y(0) + \int_0^t \sqrt{V(s)} \sum_j \sigma_j(Y(s)) dW_j(s) + \int_0^t V(s) b(Y(s)) ds$$

$$+ \int_0^t \sqrt{1 - V(s)} \sum_j s_j(Y(s)) dB_j(s) + \int_0^t (1 - V(s)) \beta(Y(s)) ds$$

As before, every solution of the controlled martingale problem corresponds to a solution of the controlled SDE, and inverting λ_0 as above,

$$X(t) = X(0) + \int_0^t \sum_j \sigma_j(X(s)) dW_j^V(s) + \int_0^t b(X(s)) ds$$

$$+ \int_0^t \sum_j s_j(X(s)) dM_j^\lambda(s) + \int_0^t \beta(X(s)) d\lambda(s),$$

Driving processes

As before,

$$W_j^V(t) = \int_0^{\tau(t)} \sqrt{V(s)} dW_j(s)$$

are independent standard Brownian motions, and now

$$M_j^\lambda(t) = \int_0^{\tau(t)} \sqrt{1 - V(s)} dB_j(s)$$

are martingales with $[M_j^\lambda]_t = \int_0^{\tau(t)} (1 - V(s)) ds = \lambda(t)$.

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Abstract

Controlled and constrained martingale problems

Most of the basic results on martingale problems extend to the setting in which the generator depends on a control. The control could represent a random environment, or the generator could specify a classical stochastic control problem. The equivalence between the martingale problem and forward equation (obtained by taking expectations of the martingales) provides the tool for extending linear programming methods introduced by Manne in the context of controlled finite Markov chains to general Markov stochastic control problems. The controlled martingale problem can also be applied to the study of constrained Markov processes (e.g., reflecting diffusions), the boundary process being treated as a control. Time permitting: the relationship between the control formulation and viscosity solutions of the corresponding resolvent equation will be discussed. Talk includes joint work with Richard Stockbridge and with Cristina Costantini.