Generators, martingale problems, and stochastic equations

kurtz@math.wisc.edu

Generators for Markov processes

An E-valued process is *Markov* wrt $\{\mathcal{F}_t\}$ if X is $\{\mathcal{F}_t\}$-adapted and

$$\mathbb{E}[f(X(t+s))|\mathcal{F}_t] = \mathbb{E}[f(X(t+s))|X(t)] \equiv T(s)f(X(t)), \quad f \in B(E)$$

$$\begin{aligned}
\mathbb{E}[f(X(t+s+r))|\mathcal{F}_t] &= T(s+r)f(X(t)) \\
&= \mathbb{E}[\mathbb{E}[f(X(t+s+r))|\mathcal{F}_{t+s}]|\mathcal{F}_t] \\
&= \mathbb{E}[T(r)f(X(t+s))|\mathcal{F}_t] \\
&= T(s)T(r)f(X(t))
\end{aligned}$$

$\{T(t), t \geq 0\}$ is a semigroup of bounded operators on $B(E)$. The *generator* for $\{T(t)\}$ satisfies

$$T(t)f = f + \int_0^t AT(s)f ds = f + \int_0^t T(s)Af ds$$

for f in a domain $\mathcal{D}(A)$.

Dynkin (1965), Ethier and Kurtz (1986)

Martingale properties

The second equality $T(t)f = f + \int_0^t T(s)Af\,ds$ can be written as

$$\begin{aligned}\mathbb{E}[f(X(r+t))|X(r)] &= \mathbb{E}[f(X(r+t))|\mathcal{F}_r] \\ &= f(X(r)) + \mathbb{E}[\int_r^{r+t} Af(X(s))ds|\mathcal{F}_r]\end{aligned}$$

which gives

$$\mathbb{E}[f(X(r+t)) - f(X(r)) - \int_r^{r+t} Af(X(s))ds|\mathcal{F}_r] = 0$$

and in turn implies

$$M_f(t) = f(X(t)) - f(X(0)) - \int_0^t Af(X(s))ds$$

is a martingale, that is $\mathbb{E}[M_f(t+r)|\mathcal{F}_r] = M_f(r)$.

This martingale property can be used to characterize the corresponding Markov process. (Stroock and Varadhan (1979))

The martingale problem for A

X is a solution for the martingale problem for (A, ν_0), $\nu_0 \in \mathcal{P}(E)$, if $PX(0)^{-1} = \nu_0$ and there exists a filtration $\{\mathcal{F}_t\}$ such that

$$f(X(t)) - f(X(0)) - \int_0^t Af(X(s))ds$$

is an $\{\mathcal{F}_t\}$-martingale for all $f \in \mathcal{D}(A)$.

Theorem 1 *If any two solutions of the martingale problem for A satisfying $PX_1(0)^{-1} = PX_2(0)^{-1}$ also satisfy $PX_1(t)^{-1} = PX_2(t)^{-1}$ for all $t \geq 0$, then the f.d.d. of a solution X are uniquely determined by $PX(0)^{-1}$*

If X is a solution of the MGP for A and $Y_a(t) = X(a+t)$, then Y_a is a solution of the MGP for A.

Theorem 2 *If the conclusion of the above theorem holds, then any solution of the martingale problem for A is a Markov process.*

Forward equations

Let ν_t be the distribution of $X(t)$ where X is a solution of the martingale problem for A. Then the fact that

$$f(X(t)) - f(X(0)) - \int_0^t Af(X(s))ds$$

is a martingale (and hence has expectation zero) implies

$$\nu_t f = \nu_0 f + \int_0^t \nu_s Af, \quad f \in \mathcal{D}(A),$$

$$\nu_t f = \int f d\nu_t$$

Of course, if A generates a semigroup,

$$\nu_t f = \nu_0 T(t) f$$

Examples of generators

Poisson process ($E = \{0, 1, 2 \ldots\}$, $\mathcal{D}(A) = B(E)$)

$$Af(k) = \lambda(f(k+1) - f(k))$$

Pure jump process (E arbitrary)

$$Af(x) = \lambda(x) \int_E (f(y) - f(x))\mu(x, dy)$$

Diffusion process ($E = \mathbb{R}^d$, $\mathcal{D}(A) = C_c^2(\mathbb{R}^d)$)

$$Af(x) = \frac{1}{2} \sum_{i,j} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} f(x) + \sum_i b_i(x) \frac{\partial}{\partial x_i} f(x)$$

ODE $\dot{X} = F(X)$ ($E = \mathbb{R}^d$, $\mathcal{D}(A) = C_c^1(\mathbb{R}^d)$)

$$Af(x) = F(x) \cdot \nabla f(x)$$

Stochastic differential equations for diffusions

$$X(t) = X(0) + \int_0^t \sigma(X(s))dW(s) + \int_0^t b(X(s))ds$$

where W is a standard Brownian motion corresponds to

$$Af(x) = \frac{1}{2}\sum_{i,j} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}f(x) + \sum_i b_i(x)\frac{\partial}{\partial x_i}f(x)$$

where $a(x) = \sigma(x)\sigma(x)^T$.

Stochastic equations for jump processes

$$X(t) = X(0) + \int_{[0,t]\times[0,\infty)\times[0,1]} \mathbf{1}_{[0,\lambda(X(s-))]}(u)(H(X(s-),v) - X(s-)) \times\xi(ds,du,dv)$$

where ξ is a Poisson random measure with mean measure $ds \times du \times dv$ (i.e., Lebesgue measure) on $[0,\infty) \times [0,\infty) \times [0,1]$. The equation corresponds to

$$Af(x) = \lambda(x) \int_E (f(y) - f(x))\mu(x,dy)$$

provided for ζ uniform $[0,1]$,

$$P\{H(x,\zeta) \in C\} = \mu(x,C)$$

Equivalence theorem: First direction

Theorem 3 *Every solution of the stochastic equation gives a solution of the martingale problem. Every solution of the martingale problem gives a solution of the forward equation.*

Proof. For example, for diffusion processes Itô's formula implies

$$f(X(t)) - f(X(0)) - \int_0^t Af(X(s))ds = \int_0^t \sigma(X(s))dW(s)$$

□

Itô for the jump process

Let $\widetilde{\xi}$ be the Poisson random measure on $[0,\infty)^2 \times [0,1]$ centered by its mean measure, that is, for $A \in \mathcal{B}([0,\infty)^2 \times [0,1]$

$$\widetilde{\xi}(A) = \xi(A) - \ell^3(A)$$

Since

$$f(X(t)) - f(X(0))$$
$$= \int_{[0,t]\times[0,\infty)\times[0,1]} \mathbf{1}_{[0,\lambda(X(s-)]}(u)(f(H(X(s-),v)) - f(X(s-)))\xi(ds,du,dv),$$

we have

$$f(X(t)) - f(X(0)) - \int_0^t \lambda(X(s))(f(y) - f(X(s))\mu(X(s),dy)$$
$$= \int_{[0,t]\times[0,\infty)\times[0,1]} \mathbf{1}_{[0,\lambda(X(s-)]}(u)(f(H(X(s-),v)) - f(X(s-)))\widetilde{\xi}(ds,du,dv)$$

Technical conditions

Condition 4 *a)* $(1,0) \in A \subset C_b(E) \times C_b(E)$

b) $\mathcal{D}(A)$ *is closed under multiplication and separates points.*

c) There exists $A_0 \subset A$ *such that* A_0 *is countable and every solution of the martingale problem for* A_0 *is a solution of the martingale problem for* A*.*

d) A *is a pre-generator, that is* A *is dissipative and for each* x *there exist* $\lambda_n^x > 0$ *and* $\mu_n^x \in \mathcal{P}(E)$ *such that for each* $(f,g) \in A$

$$g(x) = \lim_{n\to\infty} \lambda_n^x \int_E (f(y) - f(x))\mu_n^x(dy).$$

Equivalence theorem: Other direction

Theorem 5 *Suppose A satisfies Condition 4. Then every solution of the forward equation corresponds to a solution of the martingale problem and every solution of the martingale problem corresponds to a solution of the stochastic equation.*

Proof. Existence of solutions of the martingale problem corresponding to solutions of the forward equations follows from work by Echeverría (1982); Ethier and Kurtz (1986); Bhatt and Karandikar (1993); Kurtz and Stockbridge (2001).

For diffusions, existence of solutions to stochastic equations corresponding to solutions of the martingale problem was given by Stroock and Varadhan (1979). For general Markov processes in $\mathbb{R}^d$, see Kurtz (2011). For reflecting diffusions determined by submartingale problems, see Kang and Ramanan (2017). For generators given as infinite sums of bounded generators, see Etheridge and Kurtz (2018). □

Second fundamental theorem of filtering

Theorem 6 *Kurtz and Nappo (2011)*
Assume that A satisfies Condition 4. Suppose that Y is a cadlag process (sample paths are right continuous with left limits) with no fixed points of discontinuity in a complete, separable metric space V, and $\{\pi_t\}$ is a cadlag $\mathcal{P}(E)$-valued stochastic process adapted to the filtration $\{\mathcal{F}_t^Y\}$ generated by Y. If for each $f \in \mathcal{D}(A) \subset C_b(E)$

$$M_f^{\mathcal{F}^Y}(t) = \pi_t f - \pi_0 f - \int_0^t \pi_s A f ds$$

is a $\{\mathcal{F}_t^Y\}$-martingale, then there exists a filtration $\{\widetilde{\mathcal{F}}_t\}$ a process X adapted to $\{\widetilde{\mathcal{F}}_t\}$, and a $\{\widetilde{\mathcal{F}}_t\}$-adapted process $\widetilde{Y}$ such that for each $f \in \mathcal{D}(A)$

$$M_f(t) = f(X(t)) - f(X(0)) - \int_0^t Af(X(s))ds$$

is a $\{\widetilde{\mathcal{F}}_t\}$-martingale, and $(\widetilde{\pi}, \widetilde{Y})$ with $\widetilde{\pi}_t(C) = P\{X(t) \in C | \widetilde{\mathcal{F}}_t^{\widetilde{Y}}\}$, $C \in \mathcal{B}(E)$, $t \geq 0$, has the same finite dimensional distributions as (π, Y).

Equivalence of SDE and MGP

Stroock and Varadhan (1979)

If

$$X(t) = X(0) + \int_0^t \sigma(X(s))dW(s) + \int_0^t b(X(s))ds,$$

then for $f \in \mathbb{R}^d$, by Itô's formula

$$f(X(t)) - f(X(0)) - \int_0^t Af(X(s))ds = \int_0^t \nabla f(X(s))^T \sigma(X(s))dW(s)$$

for

$$Af(x) = \frac{1}{2}\sum_{ij} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}f(x) + \sum_i b_i(x)\frac{\partial}{\partial x_i}f(x)$$

where $((a_{ij})) = \sigma\sigma^T$. Consequently, X is a solution of the martingale problem for A.

Converse

If X is a solution of the MGP for A, then X is a weak solution of the SDE. If σ is invertible, then

$$W(t) = \int_0^t \sigma^{-1}(X(s))dX(s) - \int_0^t \sigma^{-1}(X(s))b(X(s))ds$$

and hence

$$\int_0^t \sigma(X(s))dW(s) = X(t) - X(0) - \int_0^t b(X(s))ds$$

Generator for general Markov process in $\mathbb{R}^d$

Assuming $S = S_1 \cup S_2$ and

$$\int_S \lambda(x,u)(\mathbf{1}_{S_1}(u)|\gamma(x,u)|^2 + \mathbf{1}_{S_2}(u))\nu(du) < \infty,$$

Then

$$\begin{aligned} Af(x) &= \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} f(x) + b(x)\cdot\nabla f(x) \\ &\quad + \int_S \lambda(x,u)(f(x+\gamma(x,u)) - f(x) - \mathbf{1}_{S_1}(u)\gamma(x,u)\cdot\nabla f(x))\nu(du). \end{aligned} \tag{1}$$

for $f \in C_c^2(\mathbb{R}^d)$.

c.f. Stroock (1975); Graham (1992); Kurtz (2011)

Stochastic equation

Let ξ be a Poisson random measure on $[0,\infty)\times S\times[0,\infty)$ with mean measure $m\times\nu\times m$, and let $\widetilde{\xi}(A)=\xi(A)-m\times\nu\times m(A)$

$$
\begin{aligned}
X(t) \;=\; & X(0)+\int_0^t \sigma(X(s))dW(s)+\int_0^t b(X(s))ds \qquad (2)\\
& +\int_{[0,1]\times S_1\times[0,t]} \mathbf{1}_{[0,\lambda(X(s-),u)]}(v)\gamma(X(s-),u)\widetilde{\xi}(dv\times du\times ds)\\
& +\int_{[0,1]\times S_2\times[0,t]} \mathbf{1}_{[0,\lambda(X(s-),u)]}(v)\gamma(X(s-),u)\xi(dv\times du\times ds),
\end{aligned}
$$

Stochastic equations of this form appeared first in Itô (1951). See also Graham (1992); Kurtz and Protter (1996); Kurtz (2011)

Alternative approach Kurtz (2011)

$$X(t) = X(0) + \int_0^t \sigma(X(s))dW(s) + \int_0^t b(X(s))ds$$

in $d = 1$. Define

$$Z(t) = Z(0) + W(t) \bmod 1,$$

where $Z(0)$ is uniformly distributed on $[0,1]$ and independent of W. Then (X,Z) is a solution of the MGP for

$$\begin{aligned}\widetilde{A}f(x,z) = \frac{1}{2}a(x)\frac{\partial^2}{\partial x^2}f(x,z) + \sigma(x)\frac{\partial^2}{\partial x\partial z}f(x,z) \\ +\frac{1}{2}\frac{\partial^2}{\partial z^2}f(x,z) + b(x)\frac{\partial}{\partial x}f(x,z),\end{aligned}$$

where we take $f \in C_c^2(\mathbb{R} \times [0,1])$ and periodic in z with period 1. If (X,Z) is a solution of the martingale problem for $\widetilde{A}$, then the corresponding W can be recovered from Z and X is a weak solution of the SDE.

Equivalence to original MGP

Let X be a solution of the martingale problem for A, and define $\overline{f}(x) = \int_0^1 f(x,z)dz$ and $\pi_t = \delta_{X(t)}(dx)dz$. Then

$$\pi_t f = \int_0^1 f(X(t),z)dz = \overline{f}(X(t))$$

and

$$\pi_t \widetilde{A} = A\overline{f}(X(t)),$$

so

$$\pi_t f - \pi_0 f - \int_0^t \pi_s \widetilde{A} f ds = \overline{f}(X(t)) - \overline{f}(X(0)) - \int_0^t A\overline{f}(X(s))ds$$

is a $\{\mathcal{F}_t^X\}$-martingale.

Construction of the weak solution

By Theorem 6, for any solution X of the MGP for A there is a solution $(\widetilde{X}, \widetilde{Z})$ of the martingale problem for $\widetilde{A}$ such that $\widetilde{X}$ has the same distribution as X and $\widetilde{Z}$ determines a Brownian motion $\widetilde{W}$ such that.

$$\widetilde{X}(t) = \widetilde{X}(0) + \int_0^t \sigma(\widetilde{X}(s))d\widetilde{W}(s) + \int_0^t b(\widetilde{X}(s))ds,$$

that is, X is a weak solution of the SDE.

Finite dimensional distributions determine a solution

X is a solution of the martingale problem for A if for each $f \in \mathcal{D}(A)$ and all choice of $h_1, \ldots, h_k \in C_b(E)$

$$E[(f(X(t_{k+1})) - f(X(t_k)) - \int_{t_k}^{t_{k+1}} Af(X(s))ds) \prod_{i=1}^{k} h_i(X(t_i))] = 0$$

for all $t_1 < t_2 < \cdots < t_k < t_{k+1}$.

Useful for convergence:

Theorem 7 *Suppose X_n is a solution of the martingale problem for A_n and for every $f \in \mathcal{D}(A)$ there exist $f_n \in \mathcal{D}(A_n)$, $f_n \to f$, $A_n f_n \to Af$, boundedly and uniformly on compacts. If $\{X_n\}$ is relatively compact in $D_E[0,\infty)$ (space of cadlag functions), then every limit point is a solution of the martingale problems for A.*

Relative compactness

Theorem 8 *Suppose E is compact (for example, $E = \mathbb{R}^d \cup \{\infty\}$) and for each f in some dense subset $\mathcal{D} \subset C(E)$ there exist $f_n \in \mathcal{D}(A_n)$ such that $f_n \to f$ uniformly and $\sup_n \|A_n f_n\| < \infty$, then $\{X_n\}$ is relatively compact in $D_E[0, \infty)$.*

Proof. The result follows from Theorems 3.9.4 and 3.9.1 of Ethier and Kurtz (1986). □

References

Abhay G. Bhatt and Rajeeva L. Karandikar. Invariant measures and evolution equations for Markov processes characterized via martingale problems. *Ann. Probab.*, 21(4):2246–2268, 1993. ISSN 0091-1798.

E. B. Dynkin. *Markov processes. Vols. I, II*, volume 122 of *Translated with the authorization and assistance of the author by J. Fabius, V. Greenberg, A. Maitra, G. Majone. Die Grundlehren der Mathematischen Wi ssenschaften, Bände 121*. Academic Press Inc., Publishers, New York, 1965.

Pedro Echeverría. A criterion for invariant measures of Markov processes. *Z. Wahrsch. Verw. Gebiete*, 61(1): 1–16, 1982. ISSN 0044-3719.

Alison Etheridge and Thomas G. Kurtz. Genealogical constructions of population models. *Ann. Probab.*, 2018. To appear.

Stewart N. Ethier and Thomas G. Kurtz. *Markov processes: Characterization and convergence*. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons Inc., New York, 1986. ISBN 0-471-08186-8.

Carl Graham. McKean-Vlasov Itô-Skorohod equations, and nonlinear diffusions with discrete jump sets. *Stochastic Process. Appl.*, 40(1):69–82, 1992. ISSN 0304-4149.

Kiyosi Itô. On stochastic differential equations. *Mem. Amer. Math. Soc.*, 1951(4):51, 1951. ISSN 0065-9266.

Weining Kang and Kavita Ramanan. On the submartingale problem for reflected diffusions in domains with piecewise smooth boundaries. *Ann. Probab.*, 45(1):404–468, 2017. ISSN 0091-1798. doi: 10.1214/16-AOP1153. URL `https://doi-org/10.1214/16-AOP1153`.

Thomas G. Kurtz. Equivalence of stochastic equations and martingale problems. In Dan Crisan, editor, *Stochastic Analysis 2010*, pages 113–130. Springer, 2011.

Thomas G. Kurtz and Giovanna Nappo. The filtered martingale problem. In Dan Crisan and Boris Rozovskii, editors, *Handbook on Nonlinear Filtering*, chapter 5, pages 129–165. Oxford University Press, 2011.

Thomas G. Kurtz and Philip E. Protter. Weak convergence of stochastic integrals and differential equations. II. Infinite-dimensional case. In *Probabilistic models for nonlinear partial differential equations (Montecatini Terme, 1995)*, volume 1627 of *Lecture Notes in Math.*, pages 197–285. Springer, Berlin, 1996.

Thomas G. Kurtz and Richard H. Stockbridge. Stationary solutions and forward equations for controlled and singular martingale problems. *Electron. J. Probab.*, 6:no. 17, 52 pp. (electronic), 2001. ISSN 1083-6489.

Daniel W. Stroock. Diffusion processes associated with Lévy generators. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 32(3):209–244, 1975.

Daniel W. Stroock and S. R. Srinivasa Varadhan. *Multidimensional diffusion processes*, volume 233 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1979. ISBN 3-540-90353-4.

Abstract

Generators, martingale problems, and stochastic equations

Classically, general Markov processes were studied through their relationship to operator semigroups. The analytic challenges of operator semigroup theory helped motivate the development of alternative approaches including stochastic equations as introduced by Ito and martingale problems as introduced by Stroock and Varadhan. These approaches have dominated work on Markov processes in the mathematics literature while the Kolmogorov forward equation that characterizes the one dimensional distributions of the process receives much more attention in the physics literature (cf. Fokker-Planck equation, master equation). The talk will include a brief over view of all these approaches paying particular attention to the equivalence of the different approaches in characterizing Markov processes.