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SEMIMARTINGALE REFLECTING BROWNIAN MOTIONS IN THE ORTHANT

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Abstract. This paper surveys recent work on semimartingale reflecting Brownian motions in the orthant. These diffusion processes have been proposed as approximate models of multiclass open queueing networks in heavy traffic. The topics covered by this survey are the problems of existence and uniqueness, recurrence classification and stationary distributions for these diffusions.

1. Introduction. Reflecting Brownian motions in the orthant have been proposed as approximate models of open queueing networks in heavy traffic (see Harrison-Nguyen [19]). Such networks are of current interest for studying congestion and delay in computer communication and manufacturing systems. For single class and feedforward multiclass networks with first-in-first-out service discipline, limit theorems to justify the diffusion approximation have been proved by Reiman [37] and Peterson [36], respectively. An outstanding open problem is to prove such a limit theorem for multiclass networks with feedback. Indeed, it has recently been discovered (see for example, Lu-Kumar [29], Rybko-Stolyar [39], Dai-Wang [11], Whitt [44], Bramson [3], [4] and Seidman [40]), that the conditions for stability of a multiclass network are not well understood. There is currently a good deal of activity directed towards resolving this problem which is a precursor to the development of a heavy traffic diffusion limit theorem for such networks. Despite the lack of a limit theorem in the multiclass case, it is still of interest and also of potential benefit for this approximation phase to develop a theory for the diffusions. This article focusses on the latter by surveying recent work on reflecting Brownian motions in the orthant. In fact, attention will be confined here to *semimartingale* reflecting Brownian motions in the orthant (SRBMs). The reasons for this are that (a) many of the queueing networks of interest lead to such processes (although there are some models of routing that lead to non-semimartingale reflecting Brownian motions — see [26] for a two-dimensional example), and (b) there currently is no theory for non-semimartingale reflecting Brownian motions in dimensions higher than two (see [6], [43], [45], [13], [46] for some results on the one and two dimensional cases).

This survey will concentrate on the following two topics for SRBMs:
(i) Synthesis — existence and uniqueness with given geometric data,
(ii) Analysis — recurrence classification and stationary distributions.

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2. Synthesis — Existence and Uniqueness. This section concerns the problem of existence and uniqueness of semimartingale reflecting Brownian motions in the orthant (SRBMs). Loosely speaking, such a process has a semimartingale decomposition such that in the interior of the orthant it behaves like a Brownian motion with a constant drift and covariance matrix, and at each of the $(d-1)$-dimensional boundary faces, the finite variation part of the process increases in a given direction (constant for a particular face), so as to confine the process to the orthant. For historical reasons, this "pushing" at the boundary is called instantaneous reflection.

For a more precise description of an SRBM, the following notation is needed. Let d be a positive integer, $S = \{x \in \mathbb{R}^d : x_i \geq 0 \text{ for } i = 1, \ldots, d\}$, θ be a constant vector in $\mathbb{R}^d$, Γ be a $d \times d$ non-degenerate covariance matrix (symmetric and positive definite), and R be a $d \times d$ matrix. A triple $(\Omega, \mathcal{F}, \{\mathcal{F}_t\})$ is called a *filtered* space if Ω is a set, $\mathcal{F}$ is a σ-field of subsets of Ω, and $\{\mathcal{F}_t\} \equiv \{\mathcal{F}_t, t \geq 0\}$ is an increasing family of sub-σ-fields of $\mathcal{F}$, i.e., a filtration.

DEFINITION 2.1. *An SRBM associated with the data (S, θ, Γ, R) is a continuous, $\{\mathcal{F}_t\}$-adapted, d-dimensional process Z together with a family of probability measures $\{P_x, x \in S\}$ defined on some filtered space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\})$ such that for each $x \in S$, under P_x,*

$$Z(t) = X(t) + RY(t) \in S \quad \textit{for all} \quad t \geq 0, \tag{2.1}$$

where

(i) *X is a d-dimensional Brownian motion with drift vector θ, covariance matrix Γ, $\{X(t) - \theta t, \mathcal{F}_t, t \geq 0\}$ is a martingale, and $X(0) = x$, P_x-a.s.,*

(ii) *Y is an $\{\mathcal{F}_t\}$-adapted, d-dimensional process such that P_x-a.s. for $i = 1, \ldots, d$,*

- **(a)** *$Y_i(0) = 0$,*
- **(b)** *Y_i is continuous and non-decreasing,*
- **(c)** *Y_i can increase only when Z is on the face $F_i \equiv \{x \in S : x_i = 0\}$, i.e., $\int_0^t 1_{F_i}(Z(s))\, dY_i(s) = Y_i(t)$ for all $t \geq 0$.*

One approach to constructing an SRBM is to try to solve the following *deterministic* Skorokhod problem for all continuous paths $x(\cdot)$ in $\mathbb{R}^d$ that start in S. This approach is based on the hope that the solutions might be given by a measurable, adapted path-to-path mapping, which could be applied to the paths of a Brownian motion to yield a "strong solution" of the equations defining an SRBM.

In the following, $\mathbf{C} \equiv C([0,\infty), \mathbb{R}^d) = \{x : [0,\infty) \to \mathbb{R}^d, \ x \text{ is continuous}\}$ and $\mathbf{C}_+ = \{x \in \mathbf{C} : x(0) \in S\}$. We endow $\mathbf{C}$ (and hence $\mathbf{C}_+$) with the topology of uniform convergence on compact time intervals and define the associated σ-fields $\mathcal{M} = \sigma\{x(s) : 0 \leq s < \infty\}$, $\mathcal{M}_t = \sigma\{x(s) : 0 \leq s \leq t\}$ for all $t \geq 0$.

DEFINITION 2.2. **(Skorokhod Problem)** Let $x \in \mathbf{C}_+$. Then $(z, y) \in \mathbf{C} \times \mathbf{C}$ solves the Skorokhod problem (SP) for x (with respect to S and R) if

(i) $z(t) = x(t) + Ry(t) \in S$ for all $t \geq 0$,
(ii) y is such that for $i = 1, \ldots, d$,
 (a) $y_i(0) = 0$,
 (b) y_i is non-decreasing,
 (c) y_i can increase only when z is on F_i.

Harrison and Reiman [20] considered this problem in connection with single class open queueing networks. The type of R matrix that arises from such networks is of the form $R = I - P'$ where P has zeros on its diagonal and is a transition matrix for a transient Markov chain on d states (corresponding to the d nodes in the network). For such an R matrix, Harrison and Reiman showed the existence of a Lipschitz continuous path-to-path mapping $\Phi : \mathbf{C}_+ \to \mathbf{C} \times \mathbf{C}$ which for each $x \in \mathbf{C}_+$ yields an adapted solution $(z, y) = \Phi(x)$ of the Skorokhod problem for x. (Here adapted means that the mapping $x \to (z_t, y_t)$ is $\mathcal{M}_t$-measurable for each $t \geq 0$.) Moreover, this solution is the unique one for x. Scrutiny of the proof in [20] reveals that the result extends to the case where $R = I - Q$, the matrix $|Q|$ (obtained by replacing each entry in Q by its absolute value) has spectral radius strictly less than one, and there are no restrictions on the signs of the entries in Q (see also Dupuis and Ishii [14]). If we apply this result to the paths of a Brownian motion then we obtain a strong solution of the SRBM equations. In summary we have the following.

THEOREM 2.1. *Suppose that $R = I - Q$ where $|Q|$ has spectral radius strictly less than one. Then for each $x \in \mathbf{C}_+$, there is a unique adapted solution $\Phi(x) = (z, y)$ of the Skorokhod problem for x. Define X, Y, Z on $\mathbf{C}_+$ by $X(x) = x$, $(Z, Y)(x) = \Phi(x)$ for each $x \in \mathbf{C}_+$, and for each $x_0 \in S$, let P_{x_0} denote the probability measure on $(\mathbf{C}_+, \mathcal{M})$ under which the canonical path $X = x(\cdot)$ is a Brownian motion with drift θ, covariance matrix Γ, and starts from x_0. Then Z together with $\{P_{x_0}, x_0 \in S\}$ is an SRBM associated with (S, θ, Γ, R). Furthermore, it defines a Feller continuous strong Markov process.*

Beyond the result of Harrison and Reiman [20], it is relatively easy to see that a necessary condition for one to be able to solve the Skorokhod problem for each $x \in \mathbf{C}_+$ is that at any point on the boundary of S, there is a positive linear combination of the vectors of reflection that one is allowed to use there, which points into the interior of the state space. This can also be shown to be necessary for the existence of an SRBM starting from each point in S (see Theorem 2.3 below). This geometric condition can be expressed succinctly as the following algebraic *completely-$\mathcal{S}$* condition on the reflection matrix R.

DEFINITION 2.3. *A principal submatrix of the $d \times d$ matrix R is any square matrix obtained from R by deleting all rows and columns from R with indices in some (possibly empty) subset of $\{1, \ldots, d\}$. The matrix R*

is completely-$\mathcal{S}$ if and only if for each principal submatrix $\tilde{R}$ of R there is $\tilde{x} \geq 0$ such that $\tilde{R}\tilde{x} > 0$.

Completely-$\mathcal{S}$ matrices are known in the operations research literature. Alternative names for these matrices are strictly semimonotone or completely-$\mathcal{Q}$ matrices. A useful property is that R is completely-$\mathcal{S}$ if and only if its transpose R' is completely-$\mathcal{S}$ (see [38], Lemma 3, p. 91).

If we denote the columns of R by $v_1, \ldots, v_d$ and the inward unit normals to the faces F_i by n_i for $i = 1, \ldots, d$, then the completely-$\mathcal{S}$ condition can be written in coordinate form as

(S.a) for each $\mathbf{K} \subset \{1, \ldots, d\}$ there is a positive linear combination $v = \sum_{i \in \mathbf{K}} a_i v_i$ ($a_i > 0\ \forall i \in \mathbf{K}$) of the $\{v_i, i \in \mathbf{K}\}$ such that $n_i \cdot v > 0$ for all $i \in \mathbf{K}$,

and its adjoint form is:

(S.b) for each $\mathbf{K} \subset \{1, \ldots, d\}$ there is a positive linear combination $n = \sum_{i \in \mathbf{K}} b_i n_i$ ($b_i > 0\ \forall i \in \mathbf{K}$) of the $\{n_i, i \in \mathbf{K}\}$ such that $n \cdot v_i > 0$ for all $i \in \mathbf{K}$.

Two sets of authors, namely, Bernard and El Kharroubi [2] and Mandelbaum and Van der Heyden [34], independently showed that there is a solution of the Skorokhod problem for all $x \in \mathbf{C}_+$ when R is completely-$\mathcal{S}$. Briefly their approach may be described as follows. Approximate the path of $x \in \mathbf{C}_+$ by a piecewise linear path $\tilde{x}$ and control this approximate path using a pushing path $\tilde{y}$ which keeps the resultant path $\tilde{z} = \tilde{x} + R\tilde{y}$ in S. Control of the affine segments of $\tilde{x}$ is achieved by solving linear complementarity problems of the following form.

DEFINITION 2.4. **(Linear Complementarity Problem)** *Let $x \in \mathbb{R}^d$. Then $(z, y) \in \mathbb{R}^d \times \mathbb{R}^d$ is a solution of the linear complementarity problem for x and R if*

(i) $z = x + Ry \in S$,

(ii) *for* $i = 1, \ldots, d$,

  (a) $y_i \geq 0$,

  (b) $z_i y_i = 0$.

One can pass to a limit in the approximation (possibly along a subsequence) using the following oscillation estimate (see Lemma 2.1) to obtain the necessary compactness. This then yields a solution of the Skorokhod problem, which is also known as the *dynamic complementarity problem.* For the statement of the oscillation estimate we need the following. For any continuous function f defined on $[t_1, t_2]$ into $\mathbf{R}^k$, some $k \geq 1$, let

$$\mathrm{Osc}(f, [t_1, t_2]) = \sup_{t_1 \leq s \leq t \leq t_2} |f(t) - f(s)|.$$

LEMMA 2.1. *Suppose that R is completely-$\mathcal{S}$. Then there is a constant κ which depends only on R, such that for any $x \in \mathbf{C}_+$ and solution (z, y) of the Skorokhod problem for x, we have for all $0 \leq t_1 < t_2 < \infty$,*

$$\mathrm{Osc}(z, [t_1, t_2]) \leq \kappa\, \mathrm{Osc}(x, [t_1, t_2]), \quad \mathrm{Osc}(y, [t_1, t_2]) \leq \kappa\, \mathrm{Osc}(x, [t_1, t_2]).$$

Proof. See Bernard and El Kharroubi [2] or Dai and Williams [12], Lemma 4.3. □

Summarizing the above we have the following.

THEOREM 2.2. *There is a solution of the Skorokhod problem for each $x \in \mathbf{C}_+$ if and only if R is completely-$\mathcal{S}$.*

The linear complementarity problem has unique solutions whenever R is a P-matrix (all principal minors are positive) (see [7]). However, solutions of the Skorokhod problem need not be unique, even if R is a P-matrix, as shown by examples of Bernard and El Kharroubi [2] and Mandelbaum [33]. This non-uniqueness has the further consequence that the authors of [2] and [34] were not able to show that there is a measurable path-to-path mapping $\Phi : \mathbf{C}_+ \to \mathbf{C} \times \mathbf{C}$ which for each $x \in \mathbf{C}_+$ yields an adapted solution $(z, y) = \Phi(x)$ of the Skorokhod problem for x. The reason for this is that in taking the limit of the approximate solutions, one may need to pass to a subsequence that depends on x, and since there is no guarantee of uniqueness, one is not able to show that the solution can be chosen to be an adapted function of x. Consequently, Theorem 2.2 cannot be used to construct strong solutions for the SRBM equations. It is still an open problem to determine a necessary and sufficient condition for the existence and uniqueness of strong solutions to the SRBM equations.

Taking a different approach, in [41], Taylor and Williams sought weak solutions of the SRBM equations. That is, rather than trying to find (Z, Y) adapted to a given X, they sought (X, Y, Z) together which satisfied the SRBM equations. This approach yielded the following weak existence and uniqueness of an SRBM.

THEOREM 2.3. *There exists an SRBM with data (S, θ, Γ, R) if and only if R is completely-$\mathcal{S}$. In this case, the SRBM is unique in law and defines a Feller continuous strong Markov process.*

Proof of Theorem 2.3. The necessity of the completely-$\mathcal{S}$ condition follows easily by considering starting points that range over all of the facets of the boundary with dimensions between zero and $d-1$ inclusive (see Reiman and Williams [38] for a proof). The sufficiency of the completely-$\mathcal{S}$ condition, as well as the uniqueness in law, is proved in Taylor and Williams [41]. An alternative method for proving existence that exploits Kurtz's [27] patchwork and constrained martingale problem methodology is used in Dai and Williams [12]. The uniqueness proof follows the general line of argument in Bass and Pardoux [1] or Kwon and Williams [28], but hinges on an ergodic property which is verified using the oscillation estimate of Lemma 2.1. The Feller continuity and strong Markov property follow by standard arguments once the existence, uniqueness and tightness of the associated probability measures are established (see [41], p. 316). □

Remark. An extension of the results in this section to semimartingale reflecting Brownian motions in convex polyhedrons has been obtained by Dai and Williams [12]. Their result is sharpest for simple convex polyhedrons. In a simple polyhedron, precisely d faces meet at a vertex in d dimensions, and then a natural generalization of the completely-$\mathcal{S}$ condition **(S.a)** is necessary and sufficient for the weak existence and uniqueness of an SRBM. For non-simple convex polyhedrons, a generalized completely-$\mathcal{S}$ condition together with an adjoint condition (cf. **(S.b)**) provides a sufficient condition for the weak existence and uniqueness of an SRBM. See [12] for more details. Reflecting Brownian motions in convex polyhedrons are of interest as approximate models of closed, capacitated and fork-join networks. In particular, the latter can lead to SRBMs in non-simple convex polyhedrons (see Nguyen [35]).

3. Analysis — Recurrence and Stationary Distributions. Henceforth we assume that the matrix R is completely-$\mathcal{S}$ and without loss of generality we assume that it has ones on its diagonal (this can always be achieved by a renormalization of the vectors of reflection). We let Z together with $\{P_x, x \in S\}$, defined on some filtered space, denote a realization of the SRBM associated with (S, θ, Γ, R). Expectations under P_x will be denoted by E_x.

We first consider the problem of determining conditions for positive recurrence of Z. If Z is positive recurrence then there is a unique stationary distribution for Z. The second part of this section is devoted to the problem of characterizing (and computing) such a stationary distribution.

DEFINITION 3.1. *The SRBM Z is positive recurrent if for each closed set A in S having positive Lebesgue measure we have $E_x[\tau_A] < \infty$ for all $x \in S$, where $\tau_A = \inf\{t \geq 0 : Z(t) \in A\}$.*

Recently Dupuis and Williams [15] proved that a sufficient condition for positive recurrence of the SRBM Z is that all solutions of a related deterministic Skorokhod problem are attracted to the origin in the following sense.

DEFINITION 3.2. *A path $z \in \mathbf{C}$ is attracted to the origin if and only if for each $\epsilon > 0$ there is $T < \infty$ such that $|z(t)| \leq \epsilon$ for all $t \geq T$.*

THEOREM 3.1. *Suppose that all solutions z of the Skorokhod problem for drift paths x of the form $x(t) = x_0 + \theta t$, $t \geq 0$, $x_0 \in S$, are attracted to the origin. Then the SRBM Z is positive recurrent and it has a unique stationary distribution.*

Proof. The details of the proof can be found in Dupuis and Williams [15]. We briefly mention some of the key aspects here. Firstly, because of the Brownian motion diffusive aspect of Z, it suffices for the positive recurrence of Z to show that $E_x[\tau_r] < \infty$ for all $x \in S$ and all r sufficiently large, where $\tau_r = \inf\{t \geq 0 : |Z(t)| \leq r\}$ (see the proof of Theorem 2.6 in [15]). The mechanism used in [15] to establish the finiteness of these first moment hitting times is to construct a Lyapunov function f that has the following properties. The function f is twice continuously differentiable on $S \setminus \{0\}$ and

(i) given $N < \infty$, there is $M < \infty$ such that $x \in S$ and $|x| \geq M$ imply $f(x) \geq N$,

(ii) given $\epsilon > 0$ there is $M < \infty$ such that $x \in S$ and $|x| \geq M$ imply $||D^2 f(x)|| \leq \epsilon$, where $||D^2 f(x)||$ denotes the matrix norm of the Hessian $D^2 f$ of f at x,

(iii) there exists $c > 0$ such that

(a) $\theta \cdot \nabla f(x) \leq -c$ for all $x \in S \setminus \{0\}$,

(b) $v_i \cdot \nabla f(x) \leq -c$ for all $x \in F_i$, $i = 1, \ldots, d$.

Here ∇ denotes the gradient of f. It follows easily from Itô's formula and the semimartingale decomposition of Z that for such a function f there is $\eta > 0$ such that for all r sufficiently large,

$$E_x[\tau_r] \leq 2f(x)/\eta.$$

The main work of [15] is in constructing the Lyapunov function f. Besides its use in proving positive recurrence, this Lyapunov function can be used to obtain bounds on moments and path excursion estimates for the SRBM. It can also be used to prove that certain functionals of processes approximating an SRBM converge weakly to the same functionals of the limit SRBM. □

Theorem 3.1 can be applied to verify the sufficiency of the following conditions for positive recurrence of an SRBM (these results were previously established by other means). It is an interesting open problem to obtain some new conditions for positive recurrence using Theorem 3.1. Another problem of interest is to determine whether the condition of Theorem 3.1 is *necessary* for positive recurrence of the SRBM.

In the following, v_{ij} denotes the jth component of the vector v_i, which in turn is the ith column of the reflection matrix R.

THEOREM 3.2. *Suppose $d = 2$. Then the SRBM Z is positive recurrent if and only if*

$$\theta_1 + v_{21}\theta_2^- < 0 \quad \textit{and} \quad \theta_2 + v_{12}\theta_1^- < 0,$$

where the minus sign superscript denotes the negative part of a number and since R has ones on its diagonal, $v_{11} = 1, v_{22} = 1$.

Proof. See Hobson and Rogers [24] for the case $\theta \neq 0$, and Williams [46] for $\theta = 0$. The sufficiency of the above conditions can also be verified by hand using Theorem 3.1. □

THEOREM 3.3. *Suppose $R = I - P'$ where P has zeros on its diagonal and is a transition matrix for a transient Markov chain on d states. Then the SRBM Z is positive recurrent if and only if $R^{-1}\theta < 0$, where the inequality is understood to hold component by component.*

Proof. See Harrison and Williams [21]. The necessity of the condition is easy to establish. An alternative proof of the sufficiency can be given using Theorem 3.1 and the stability of solutions of the Skorokhod problem for drift paths established by Chen and Mandelbaum [5], Theorem 5.2. □

Remark. Stimulated by the result in Theorem 3.1, J. G. Dai [8] has proved an analogue for queueing networks, namely that stability of fluid limits associated with a queueing network implies positive recurrence for a Markov process that describes the dynamics of the network. Dai's method of proof is different from that in [15] in the sense that he does not construct a Lyapunov function. See other papers in this volume for discussion of how Dai's result can be used to determine sufficient conditions for the stability of multiclass networks with feedback.

Remark. Malyshev and his collaborators [30], [32], [31], [25], have been working on problems of recurrence for reflected random walks in orthants. They too use a related deterministic dynamical system and Lyapunov functions to obtain conditions for positive recurrence of their reflected random walks. However, the details of how they obtain their deterministic dynamical system from the reflected random walk seem to be different from how the Skorokhod problem with drift paths is related to the SRBM. It would be interesting to further investigate the connections and differences between these approaches.

Let us now turn to the problem of characterizing the stationary distribution of the SRBM Z.

DEFINITION 3.3. *A stationary distribution for Z is a probability measure π on the state space S, endowed with the Borel σ-algebra, such that for each real-valued, bounded Borel measurable function f defined on S, we have*

$$\int_S E_x[f(Z(t))]\,\pi(dx) = \int_S f(x)\,\pi(dx) \quad \text{for all } t \geq 0.$$

For such a π, we shall let $P_\pi = \int_S P_x \pi(dx)$ and E_π denote expectation under P_π. Parts (i) and (ii) of the following result were proved in Harrison and Williams [21] for the case where $R = I - P'$, P has zeros on the diagonal and is the transition matrix for a transient Markov chain. However, as described in Dai and Harrison [9], the proofs easily extend to the case where R is completely-$\mathcal{S}$. A simple proof of part (iii) is given in [9].

LEMMA 3.1. *Suppose that Z has a stationary distribution π. Then the following hold.*

(i) *The stationary distribution π is unique and it has a density p_0 relative to Lebesgue measure on S.*

(ii) *For each $i \in \{1, \dots, d\}$, there exists a finite Borel measure ν_i on F_i that has a density p_i relative to surface measure σ_i on F_i, and*

$$E_\pi\left[\int_0^t f(Z(s))\,dY_i(s)\right] = t\int_{F_i} f(x)\nu_i(dx),$$

for all real-valued, bounded Borel measurable functions f defined on F_i.

(iii) *The matrix R is invertible.*

From the abstract theory of Markov processes, we know that a stationary distribution for Z can be characterized by the property that it annihilates all of the functions in the range of the infinitesimal generator (see [16] for example), i.e., $\int_S f d\pi = 0$ for all f in the range of the infinitesimal generator. However, in general, for multidimensional diffusions with boundary and especially for the case treated here where the boundary conditions are non-smooth, it is virtually impossible to characterize the range of the generator. Thus, it is natural to seek an alternative annihilation characterization. The one to be given here has the virtue that it incorporates boundary information into the annihilation relation, rather than incorporating it by restricting to a class of test functions f that satisfy certain boundary conditions. To obtain the characterization we need to make a connection between the process Z and its associated analytical theory. Since Z is a semimartingale, the natural mechanism for this is Itô's formula. Let $C_b^2(S)$ denote the space of continuous, bounded, real-valued functions defined on S that have bounded continuous first and second partial derivatives on S, and suppose that the semimartingale decomposition of Z is given by (2.1). Then, for each $f \in C_b^2(S)$ and $x \in S$, we have P_x-a.s. for all $t \geq 0$,

$$
\begin{aligned}
f(Z(t)) - f(Z(0)) &= \int_0^t \nabla f(Z(s)) \cdot dB(s) \\
&\quad + \sum_{i=1}^{d} \int_0^t D_i f(Z(s))\, dY_i(s) \\
&\quad + \int_0^t Lf(Z(s))\, ds,
\end{aligned}
\tag{3.1}
$$

where the first integral is a stochastic integral with respect to the driftless Brownian motion $B(t) = X(t) - \theta t$,

$$
Lf = \frac{1}{2} \sum_{i,j=1}^{d} \Gamma_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_{i=1}^{d} \theta_i \frac{\partial f}{\partial x_i},
$$

and

$$
D_i f = v_i \cdot \nabla f, \quad \text{for } \; i = 1, \dots, d.
$$

Suppose π is a stationary distribution for Z. Taking expectations in (3.1) under E_π and using Lemma 3.1 we see that $p = (p_0; p_1, \dots, p_d)$ must satisfy the following *basic adjoint relation:*

$$
\int_S Lf\, p_0\, dx + \sum_{i=1}^{d} \int_{F_i} D_i f\, p_i\, d\sigma_i = 0 \quad \text{for all } \; f \in C_b^2(S).
\tag{3.2}
$$

Thus, (3.2) is a necessary condition that must be satisfied by the density p associated with a stationary distribution π and the auxiliary measures

ν_i, $i = 1, \ldots, d$. In fact, (3.2) characterizes the stationary distribution, as shown by the following theorem of Dai and Kurtz [10].

THEOREM 3.4. *Suppose that p_0 is a probability density on S (relative to Lebesgue measure) and for each $i \in \{1, \ldots, d\}$, p_i is a non-negative integrable (with respect to σ_i) Borel measurable function defined on F_i. If $p = (p_0; p_1, \ldots, p_d)$ satisfies the basic adjoint relation (3.2), then p_0 is the stationary density for Z and $d\nu_i = p_i d\sigma_i$ defines the boundary measures described in Lemma 3.1(ii).*

We may think of the integral relation (3.2) as the weak form of an elliptic partial differential equation with oblique derivative boundary conditions. One might be tempted to try to work with these differential equations directly, rather than with the integral relation (3.2). However, there is a primary difficulty with this. From the probabilistic derivation of (3.2), we do not obtain any smoothness properties of p other than the obvious application of Weyl's lemma which yields that p_0 is C^∞ in the interior of S. In particular, it is difficult to determine the regularity properties of p_0 near the boundary of S. Furthermore, experience with closed form solutions for two dimensional cases (see [17], [18], [42]), suggests that p_0 may have singularities at some of the non-smooth parts of the boundary. Direct analysis of (3.2) seems to be more fruitful. Indeed, we have the following result on product form solutions.

DEFINITION 3.4. *The SRBM Z is said to have a product form stationary distribution if it has a stationary density p_0 which can be written in the form*

$$p_0(x) = \prod_{i=1}^{d} p_0^i(x_i) \quad \textit{for all } x \in S,$$

where for each $i \in \{1, \ldots, d\}$, p_0^i is a probability density relative to Lebesgue measure on $[0, \infty)$.

THEOREM 3.5. *The SRBM Z has a product form stationary distribution if and only if*

$$\gamma = -R^{-1}\theta > 0$$

and

$$2\Gamma = R\Lambda + \Lambda R'$$

where Λ is a diagonal matrix with the same diagonal entries as Γ. In this case,

$$p_0(x) = C \exp(-\eta \cdot x) \quad \textit{for all } x \in S,$$

where $\eta = 2\Lambda^{-1}\gamma$, $C = \prod_{i=1}^{d} \eta_i$, and the boundary density p_i is the restriction of $\frac{1}{2}\Gamma_{ii} p_0$ to F_i, $i = 1, \ldots, d$.

Proof. As observed by Dai and Harrison [9], the proof given in Harrison and Williams [21] for the case of matrices R that come from single class open queueing networks extends to all completely-$\mathcal{S}$ matrices R. □

For matrices R that are associated with Brownian models of feedforward (multiclass) networks, an interpretation of the above product form condition was given in Harrison and Williams [23], in terms of a notion of quasireversibility for the Brownian model. Furthermore, examples were given in [23] of queueing networks which are not known to be of product form, but for which the approximating Brownian model has a product form stationary distribution.

In general, it is unlikely that one will be able to find closed form solutions to (3.2). Thus one is naturally led to consider numerical methods. Dai and Harrison [9] have initiated work in this direction by using an L^2 projection scheme to find approximate solutions of this relation with $p_i = p_0/2$. One of the interesting problems in this area is that it is difficult to impossible to test numerically whether a function is positive. Thus, one would like to have a characterization of the stationary density as in Theorem 3.4, but without the positivity assumptions on $p_0, p_1, \ldots, p_d$.

4. Conclusion. As the preceding survey shows, a good deal of progress has been made on the theory of reflecting Brownian motions in polyhedral domains in recent years. However, a number of open problems remain. A selection of these is summarized below.

(i) To prove a heavy traffic limit theorem for multiclass open queueing networks which justifies the use of SRBMs as approximate models for such.

(ii) To determine necessary and sufficient conditions for the existence and uniqueness of semimartingale reflecting Brownian motions in nonsimple convex polyhedrons (only sufficient conditions are given in Dai-Williams [12]).

(iii) Develop a theory of non-semimartingale reflecting Brownian motions in convex polyhedrons in three and more dimensions.

(iv) Use the sufficient condition of Theorem 3.1 to obtain concrete algebraic conditions for the positive recurrence of SRBMs. Also, determine whether this condition is necessary.

(v) Try to free the analytic characterization (cf. Theorem 3.4) of the stationary density for an SRBM from the a priori assumption of positivity of the densities $p_0, p_1, \ldots, p_d$. Also, try to develop new numerical schemes for computing such stationary densities or related moments.

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