Invariant States and Rates of Convergence for a Critical Fluid Model of a Processor Sharing Queue

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Abstract

This paper contains an asymptotic analysis of a fluid model for a heavily loaded processor sharing queue. Specifically, we consider the behavior of solutions of critical fluid models as time approaches infinity. The main theorems of the paper provide sufficient conditions for a fluid model solution to converge to an invariant state and, under slightly more restrictive assumptions, provide a rate of convergence. These results are used in a related work by Gromoll [8] for establishing a heavy traffic diffusion approximation for a processor sharing queue.

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Contents

1 Introduction

This paper is a sequel to [9], which establishes a fluid (or functional law of large numbers) approximation for a heavily loaded processor sharing queue. In [9], a stochastic process $\mu(\cdot)$ taking values in $\mathcal{M}_F$, the space of finite, nonnegative Borel measures on $\mathbb{R}_+ = [0, \infty)$ endowed with the topology of weak convergence, is used to track the evolution in time of the state of a processor sharing queue. At time t, $\mu(t)$ is the measure that has one unit of mass at the residual service time of each job present in the system at time t. From the *measure valued state descriptor* $\mu(\cdot)$, one can recover the traditional performance processes, such as the queue length and workload processes (cf. [9], Section 2.3). Under mild conditions, it is proved in [9] that the fluid scaled state descriptors for a sequence of heavily loaded processor sharing queues converge in distribution to a measure valued stochastic process, which we refer to as a *fluid limit* (cf. [9], Theorem 3.2). Almost every sample path of this fluid limit is a solution of a certain (deterministic) critical fluid model. In this paper, we study the asymptotic behavior as time tends to infinity of the solutions of this critical fluid model.

In [1, 2], Bramson studied the asymptotic behavior of solutions of critical fluid models associated with open multiclass queueing networks operating under two HL (head-of-the-line) service disciplines. Then, in [3], Bramson showed that, if the critical fluid model associated with an open multiclass HL queueing network has a certain asymptotic property, then a condition known as state space collapse holds. In a companion work to [3], Williams [13] showed that state space collapse, plus an algebraic condition on the first order queueing model data, is sufficient to imply a heavy traffic diffusion approximation for an open multiclass queueing network operating under a HL service discipline. To illustrate this modular approach, Bramson [3] and Williams [13] applied their results, together with the results of Bramson [1, 2], to obtain new heavy traffic diffusion limit theorems for FIFO networks of Kelly type and for networks with a HLPPS (head-of-the-line proportional processor sharing) service discipline. Processor sharing, as considered in this paper, is not a HL service discipline. However, an analogue of the modular approach of [3, 13] is developed for a processor sharing queue in Gromoll [8]. The results proved here are used in [8] to prove a state space collapse result, which in turn is used in [8] to establish a heavy traffic diffusion approximation for a processor sharing queue.

In order to state our results, we need to recall the description of the *critical fluid model* from [9]. The model has two parameters, $\alpha \in (0, \infty)$ and a Borel probability measure ν on $\mathbb{R}_+$ that does not charge the origin ($\nu(\{0\}) = 0$) and has a finite first moment ($\int_{\mathbb{R}_+} x\nu(dx) < \infty$). These parameters correspond to parameters in the queueing system. Specifically, α corresponds to the long run average rate at which jobs arrive to the system, and the probability measure ν corresponds to the distribution of the i.i.d. service times for those jobs. The qualifier *critical* refers to the fact that we are interested in the critically loaded regime where the service and arrival rates are equal. Thus, it is assumed throughout that

$$\alpha = \left(\int_{\mathbb{R}_+} x\nu(dx) \right)^{-1}. \tag{1.1}$$

The pair (α, ν) is referred to as the data for the critical fluid model, or simply the *critical data.* Here we only consider solutions of a critical fluid model, and we simply refer to these as fluid model solutions. In particular, the assumption of critical data is implicit.

A fluid model solution $\bar{\mu}(\cdot)$ is a deterministic function of time, taking values in $\mathcal{M}_F$, that satisfies conditions (C.1)–(C.4) below. In order to state these conditions, we need to introduce some notation. For a Borel set $A \subset \mathbb{R}_+$, let 1_A denote the indicator function of the set A. To ease the notation, we use the shorthand notation 1 in place of $1_{\mathbb{R}_+}$. For $\zeta \in \mathcal{M}_F$, the real valued projection of ζ associated with a bounded, real valued, Borel measurable function g defined on $\mathbb{R}_+$

is denoted by $\langle g, \zeta \rangle = \int_{\mathbb{R}_+} g(x)\zeta(dx)$. The conditions that a $\mathcal{M}_F$ valued function $\bar{\mu}(\cdot)$ must satisfy in order to be a fluid model solution involve the real valued projections of $\bar{\mu}(\cdot)$ over the class of functions

$$\mathcal{C} = \{g \in \mathbf{C}_b^1(\mathbb{R}_+) : g(0) = 0, g'(0) = 0\}.$$

Here, $\mathbf{C}_b^1(\mathbb{R}_+)$ denotes the space of once continuously differentiable real valued functions defined on $\mathbb{R}_+$ that, together with their first derivatives, are bounded on $\mathbb{R}_+$. The requirement that g and g' vanish at the origin is imposed to avoid singular behavior of $\langle g, \bar{\mu}(\cdot)\rangle$ and $\langle g', \bar{\mu}(\cdot)\rangle$, associated with mass in the fluid model abruptly disappearing as it reaches the origin. Such behavior corresponds to jobs in the queueing system abruptly departing when their residual service times reach zero.

A *fluid model solution* is a function $\bar{\mu} : [0, \infty) \longrightarrow \mathcal{M}_F$ that satisfies the following four conditions.

(C.1) The function $\bar{\mu}(\cdot)$ is continuous.

(C.2) For each $t \geq 0$, $\langle 1_{\{0\}}, \bar{\mu}(t) \rangle = 0$.

(C.3) For each $g \in \mathcal{C}$, $\bar{\mu}(\cdot)$ satisfies

$$\langle g, \bar{\mu}(t)\rangle = \langle g, \bar{\mu}(0)\rangle - \int_0^t \frac{\langle g', \bar{\mu}(s)\rangle}{\langle 1, \bar{\mu}(s)\rangle} ds + \alpha t \langle g, \nu \rangle, \tag{1.2}$$

for all $0 \leq t < t^* = \inf\{s \geq 0 : \langle 1, \bar{\mu}(s)\rangle = 0\}$.

(C.4) For all $t \geq t^*$, $\langle 1, \bar{\mu}(t)\rangle = 0$.

See [9], Section 3.1 for an interpretation of (C.1)–(C.4).

To facilitate the present discussion, we review some results from [9] concerning fluid model solutions. Let

$$\mathcal{M}_F^c = \{\xi \in \mathcal{M}_F : \langle 1_{\{x\}}, \xi \rangle = 0 \text{ for all } x \in \mathbb{R}_+\},$$

where c stands for continuous. Theorem 3.1 in [9] states that for each measure $\xi \in \mathcal{M}_F^c$, there exists a unique fluid model solution $\bar{\mu}_\xi(\cdot)$ such that $\bar{\mu}_\xi(0) = \xi$. If $\xi = \mathbf{0}$, where $\mathbf{0}$ denotes the zero measure, then, by (C.4), $\bar{\mu}_\xi(\cdot) \equiv \mathbf{0}$. Let

$$\mathcal{M}_F^{c,p} = \{\xi \in \mathcal{M}_F^c : \xi \neq \mathbf{0}\},$$

where p stands for positive. In [9], it was also shown that for $\xi \in \mathcal{M}_F^{c,p}$, $\bar{\mu}_\xi(t) \in \mathcal{M}_F^{c,p}$ for all $t \geq 0$ (cf. [9], Theorem 3.1 and Proposition 4.6). In particular, if $\xi \in \mathcal{M}_F^{c,p}$, then $t^* = \infty$.

Given $\xi \in \mathcal{M}_F^c$, it is natural to ask about the asymptotic behavior as t tends to infinity of $\bar{\mu}_\xi(t)$. Specifically, as t tends to infinity, does $\bar{\mu}_\xi(t)$ converge in some sense? If so, what is the limit and how fast is the convergence? To answer these questions, we begin by identifying the possible limiting measures. Extending the terminology in [3] to the present setting, a measure $\xi \in \mathcal{M}_F^c$ is said to be an *invariant state* if

$$\bar{\mu}_\xi(t) = \xi, \qquad \text{for all } t \geq 0.$$

Similarly, the collection of invariant states $\mathcal{I}$, which is given by

$$\mathcal{I} = \{\xi \in \mathcal{M}_F^c : \bar{\mu}_\xi(t) = \xi \text{ for all } t \geq 0\},$$

is called the *invariant manifold.* Here, it turns out that the invariant manifold $\mathcal{I}$ is a one parameter family of measures that is determined by the probability measure ν. To describe $\mathcal{I}$, we need to introduce some notation. Let F denote the cumulative distribution function associated with the probability measure ν. The distribution function F has associated with it an excess lifetime cumulative distribution function F_e, which is given by

$$F_e(x) = \alpha \int_0^x (1 - F(y))\, dy, \qquad \text{for all } x \in \mathbb{R}_+.$$

In particular, F_e has probability density function

$$f_e(x) = \alpha\,(1 - F(x))\,, \qquad \text{for all } x \in \mathbb{R}_+.$$

Note that (1.1) was used to simplify the form of the normalizing constant here. Let ν_e denote the Borel probability measure on $\mathbb{R}_+$ that has density function f_e, i.e., $\langle 1_{[0,x]}, \nu_e \rangle = F_e(x) = \int_0^x f_e(y)dy$ for all $x \in \mathbb{R}_+$. We call ν_e the *excess lifetime probability measure.* Define

$$\beta_e = \frac{1}{\langle \chi, \nu_e \rangle},$$

which is interpreted as zero if the first moment of ν_e is infinite.

Theorem 1.1 *A measure $\xi \in \mathcal{M}_F^c$ is an invariant state if and only if $\xi = c\nu_e$, for some $c \in [0, \infty)$. Equivalently, the invariant manifold $\mathcal{I}$ is given by*

$$\mathcal{I} = \{c\nu_e : c \in [0, \infty)\}.$$

Theorem 1.1 is proved in Section 3.

Let $\xi \in \mathcal{M}_F^c$. We wish to identify conditions under which $\bar{\mu}_\xi(t)$ converges to a point on the invariant manifold as t tends to infinity, and to determine the limiting state. For this, we recall that for each $t \geq 0$ the fluid analogue of the workload at time t is given by $\langle \chi, \bar{\mu}_\xi(t) \rangle$, where $\chi(x) = x$ for all $x \in \mathbb{R}_+$. By Theorem 3.1 in [9], $\langle \chi, \bar{\mu}_\xi(t) \rangle = \langle \chi, \xi \rangle$ for all $t \geq 0$. (This holds even if $\langle \chi, \xi \rangle = \infty$, in which case $\langle \chi, \bar{\mu}_\xi(t) \rangle = \infty$ for all $t \geq 0$.) Thus, if $\bar{\mu}_\xi(t)$ converges to an element $c\nu_e$ in $\mathcal{I}$ as t tends to infinity and both $\langle \chi, \xi \rangle$ and $\langle \chi, \nu_e \rangle$ are finite, then one might expect the first moment, $c \langle \chi, \nu_e \rangle$, of the limit to be given by $\langle \chi, \xi \rangle$, or equivalently that $c = \beta_e \langle \chi, \xi \rangle$. Indeed, we have the following.

Theorem 1.2 *Let $\xi \in \mathcal{M}_F^c$. If $\langle \chi, \xi \rangle < \infty$, then $\bar{\mu}_\xi(t)$ converges weakly to $\beta_e \langle \chi, \xi \rangle \nu_e$ as $t \to \infty$.*

Notice that, $\beta_e > 0$ if and only if $\langle \chi^2, \nu \rangle < \infty$, since, for $\gamma \in \mathbb{R}_+$, $\langle \chi^\gamma, \nu_e \rangle < \infty$ if and only if $\langle \chi^{\gamma+1}, \nu \rangle < \infty$. Therefore, the case in which $\xi \neq \mathbf{0}$ and $\langle \chi^2, \nu \rangle = \infty$ is degenerate in the sense that $\bar{\mu}_\xi(t)$ converges to the zero measure as t tends to infinity, but $\langle \chi, \bar{\mu}_\xi(t) \rangle$ does not converge to zero. Theorem 1.2 is proved in Section 4.

Finally, we wish to give a rate at which $\bar{\mu}_\xi(t)$ converges as t tends to infinity. In fact, we will prove two rate of convergence results. The first gives a rate of convergence in terms of a metric on $\mathcal{M}_F$ that induces the weak topology. For this, let ρ denote the extension of the Prohorov metric to $\mathcal{M}_F$. Specifically, for $\zeta_1, \zeta_2 \in \mathcal{M}_F$, $\rho(\zeta_1, \zeta_2)$ is given by

$$\begin{aligned}\rho(\zeta_1, \zeta_2) = \inf\{\delta > 0 : \langle 1_B, \zeta_1 \rangle \leq \langle 1_{B^\delta}, \zeta_2 \rangle + \delta \quad \text{and} \quad \langle 1_B, \zeta_2 \rangle \leq \langle 1_{B^\delta}, \zeta_1 \rangle + \delta, \\ \text{for all nonempty, closed sets } B \subset \mathbb{R}_+\},\end{aligned} \tag{1.3}$$

where, for each nonempty, closed set $B \subset \mathbb{R}_+$,

$$B^\delta = \{x \in \mathbb{R}_+ : \inf_{y \in B} |x - y| < \delta\}.$$

Note that, under ρ, $\mathcal{M}_F$ is a Polish space. Moreover, if $\{\zeta_n, n = 1, 2, \dots\} \subset \mathcal{M}_F$ and $\zeta \in \mathcal{M}_F$, then ζ_n converges weakly to ζ as n tends to infinity if and only if $\lim_{n\to\infty} \rho(\zeta_n, \zeta) = 0$ (cf. [5], Chapter 3, Theorems 1.7 and 3.1, which readily generalize from the set of Borel probability measures to $\mathcal{M}_F$). Our second rate result gives a rate of convergence in terms of the total variation distance. For a signed, Borel measure ζ on $\mathbb{R}_+$,

$$\|\zeta\|_{\mathrm{TV}} = \sup\{|\langle g, \zeta \rangle| \text{ such that } g : \mathbb{R}_+ \to \mathbb{R} \text{ is Borel measurable and } |g(x)| \leq 1 \text{ for all } x \in \mathbb{R}_+\}. \tag{1.4}$$

Note that, if $\{\zeta_n, n = 1, 2, \dots\} \subset \mathcal{M}_F$, $\zeta \in \mathcal{M}_F$, and $\lim_{n\to\infty} \|\zeta_n - \zeta\|_{\mathrm{TV}} = 0$, then ζ_n converges weakly to ζ. However, in general, the converse is not true (cf. [4], page 69). For each of our rate of convergence results, the convergence is uniform over sets of initial conditions satisfying certain moment constraints. These sets take the following form. For any finite, positive constants ϵ and M, let

$$\mathcal{B}_{\rho}^{M,\epsilon} = \{\xi \in \mathcal{M}_F^c : \langle 1, \xi\rangle \vee \langle \chi, \xi\rangle \vee \langle \chi^{1+\epsilon}, \xi\rangle \le M\}, \tag{1.5}$$

$$\mathcal{B}_{\mathrm{TV}}^{M,\epsilon} = \{\xi \in \mathcal{M}_F^c : \langle 1, \xi\rangle \vee \langle \chi, \xi\rangle \vee \langle \chi^2, \xi\rangle \vee \langle \chi^{2+\epsilon}, \xi\rangle \le M\}. \tag{1.6}$$

Our rate of convergence results are summarized by the following.

Theorem 1.3 *Let M be a fixed, finite, positive constant.*

(i) If, for some $\epsilon > 0$, $\langle \chi^{2+\epsilon}, \nu\rangle < \infty$, then there exists a finite, positive constant C_ρ, and a finite, positive time T_ρ, such that

$$\sup_{\xi \in \mathcal{B}_{\rho}^{M,\epsilon}} \rho\left(\bar{\mu}_\xi(t), \beta_e \langle \chi, \xi\rangle \nu_e\right) \le C_\rho t^{-\epsilon/4}, \qquad \text{for all } t \ge T_\rho. \tag{1.7}$$

(ii) If, for some $\epsilon > 0$, $\langle \chi^{3+\epsilon}, \nu\rangle < \infty$, then there exists a finite, positive constant C_{TV}, and a finite, positive time T_{TV}, such that

$$\sup_{\xi \in \mathcal{B}_{\mathrm{TV}}^{M,\epsilon}} \|\bar{\mu}_\xi(t) - \beta_e \langle \chi, \xi\rangle \nu_e\|_{\mathrm{TV}} \le C_{\mathrm{TV}} t^{-\epsilon}, \qquad \text{for all } t \ge T_{\mathrm{TV}}. \tag{1.8}$$

In Theorem 1.3, the times T_ρ and T_{TV} and the constants C_ρ and C_{TV} depend on the values of the constants M and ϵ, and on the critical data (α, ν). In the proofs, we have not tried to obtain the best possible estimates for these constants.

In [8], Theorem 1.3(i) is used to prove a state space collapse result. In that application, it is the uniform convergence over compact sets of the form $\mathcal{B}_{\rho}^{M,\epsilon}$ for $M, \epsilon \in (0, \infty)$ that is critical. In fact, the specific rate and value of the constants are not important for the argument. Although Theorem 1.3(ii) is not needed for [8], we have included it here for its intrinsic interest and potential use in other applications.

The proof of part (i) of Theorem 1.3 exploits the asymptotic behavior of the renewal function for a zero-delayed renewal process with interarrival distribution determined by the probability measure ν_e. The condition $\langle \chi^{2+\epsilon}, \nu\rangle < \infty$ is slightly stronger than requiring that this interarrival distribution has a finite mean, i.e., that $\beta_e > 0$. This condition is used in the proof to obtain a rate of convergence for Blackwell's renewal theorem. Similarly, the proof of part (ii) of Theorem 1.3 exploits the asymptotic behavior of the renewal measures for certain delayed renewal processes with interarrival distribution determined by the probability measure ν_e. The condition $\langle \chi^{3+\epsilon}, \nu \rangle < \infty$ is slightly stronger than requiring that the interarrival distribution has a finite second moment. This condition is used in the proof to obtain a rate at which the renewal measures converge in the total variation distance to the stationary renewal measure. Both the rate of convergence for Blackwell's renewal theorem and the rates of convergence for renewal measures rely on coupling results developed in [11]. In using those results, we pay careful attention to the dependence of the various constants on the initial measure ξ and the interarrival distribution ν_e.

The remainder of the paper is devoted to the proofs of Theorems 1.1, 1.2 and 1.3. Section 2 contains some background and two preparatory lemmas. Then Theorems 1.1, 1.2, 1.3(i), and 1.3(ii) are proved in Sections 3, 4, 5, and 6 respectively. In the Appendix, coupling results from [11] are applied to verify some of the estimates used in Section 6 to prove Theorem 1.3(ii).

2 Background

Recall that, for $\xi \in \mathcal{M}_F^c$, $\bar{\mu}_\xi(\cdot)$ denotes the unique fluid model solution such that $\bar{\mu}_\xi(0) = \xi$. Given $\xi \in \mathcal{M}_F^c$, the fluid analogue of the queue length $\bar{Z}(\cdot)$ is defined by

$$\bar{Z}(t) = \langle 1, \bar{\mu}_\xi(t) \rangle, \qquad \text{for all } t \geq 0. \tag{2.1}$$

For obvious reasons, $\bar{Z}(t)$ is referred to as the *total mass* at time t. Due to (C.1), $\bar{Z}(\cdot)$ is continuous. As previously noted, if $\xi \neq \mathbf{0}$, then $\bar{\mu}_\xi(t) \neq \mathbf{0}$ for all $t \geq 0$ (cf. [9], Theorem 3.1), and so $\bar{Z}(t)$ is strictly positive for all $t \geq 0$. Conversely, if $\xi = \mathbf{0}$, then $\bar{\mu}_\xi(\cdot) \equiv \mathbf{0}$ and $\bar{Z}(\cdot) \equiv 0$. Given this, for each $t \geq 0$, the fluid analogue of the cumulative service per job $\bar{S}(t)$ is defined by

$$\bar{S}(t) = \begin{cases} 0, & \text{if } \xi = \mathbf{0}, \\ \int_0^t \left(\bar{Z}(s)\right)^{-1} ds, & \text{otherwise.} \end{cases} \tag{2.2}$$

Thus, at time $t \geq 0$, $\bar{S}(t)$ denotes the *cumulative service per unit of mass* in the system up to time t. Since $\bar{Z}(\cdot)$ is continuous, $\bar{S}(\cdot)$ is continuously differentiable. The reader will note that, in order to avoid cluttering the notation, we choose not to append a subscript ξ to quantities defined by (2.1) and (2.2), since it is typically clear from the context which fluid model solution is under consideration. In [9], it was shown that, if $\xi \in \mathcal{M}_F^{c,p}$, then, for each $t \geq 0$ and $x \in \mathbb{R}_+$,

$$\left\langle 1_{[0,x]}, \bar{\mu}_\xi(t) \right\rangle = \left\langle 1_{(\bar{S}(t),\bar{S}(t)+x]}, \xi \right\rangle + \int_0^t G^x(\bar{S}(t) - \bar{S}(s))ds, \tag{2.3}$$

where, for each $x \in \mathbb{R}_+$,

$$G^x(y) = f_e(y) - f_e(x+y), \qquad \text{for all } y \in \mathbb{R}_+,$$

(cf. [9], Lemma 4.3 and (4.33)). Here we have used (C.2) above. For each $t \geq 0$, this gives an explicit description of the measure $\bar{\mu}_\xi(t)$ in terms of the nonzero initial measure ξ and the cumulative service per unit of mass function $\bar{S}(\cdot)$.

In order to state what is known about $\bar{S}(\cdot)$ for a given $\xi \in \mathcal{M}_F^{c,p}$, we need to introduce the *renewal function* $U_e(\cdot)$ associated with the critical data (α, ν) and the *truncated initial workload function* $H_\xi(\cdot)$ associated with an initial measure ξ. For this, given a locally bounded, Borel measurable function $g : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ and a right continuous function $U : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ that is locally of bounded variation, let

$$(g * U)(u) = \int_{[0,u]} g(u-s)dU(s), \qquad \text{for all } u \geq 0.$$

Note that, by convention, the contribution to the above integral is $g(u)U(0)$ at $s = 0$ whenever $U(0) \neq 0$. Let

$$U_e(u) = \sum_{i=0}^{\infty} \left(F_e^{*i}\right)(u), \qquad \text{for all } u \geq 0,$$

where $F_e^{*0}(\cdot) \equiv 1$ and $F_e^{*i}(\cdot) = \left(F_e^{*(i-1)} * F_e\right)(\cdot)$, for each $i \in \{1, 2, \dots\}$. For $\xi \in \mathcal{M}_F^{c,p}$, define

$$H_\xi(x) = \int_0^x \left\langle 1_{(y,\infty)}, \xi \right\rangle dy, \qquad \text{for all } x \in \mathbb{R}_+. \tag{2.4}$$

It is easily verified that, for each $x \in \mathbb{R}_+$, $H_\xi(x) = \langle \chi \wedge x, \xi \rangle$, which explains why H_ξ is referred to as the truncated initial workload function. Since ξ has no atoms, the integrand in (2.4) is continuous. Thus, $H_\xi(\cdot)$ is continuously differentiable with

$$H_\xi'(x) = \left\langle 1_{(x,\infty)}, \xi \right\rangle, \qquad \text{for all } x \in \mathbb{R}_+. \tag{2.5}$$

In [9], it was shown that, for $\xi \in \mathcal{M}_F^{c,p}$, $\bar{S}(t) = \bar{T}^{-1}(t) = \inf\{u \geq 0 : \bar{T}(u) > t\}$ for all $t \geq 0$, where

$$\bar{T}(u) = (H_\xi * U_e)(u), \qquad \text{for all } u \geq 0, \tag{2.6}$$

(cf. [9], Lemma 4.4). Moreover, $\bar{T}(\cdot)$ is continuously differentiable with

$$\bar{T}'(u) = \left(H_\xi' * U_e\right)(u), \qquad \text{for all } u \geq 0. \tag{2.7}$$

In the next lemma, we use the fact that $\bar{S}(\cdot) = \bar{T}^{-1}(\cdot)$ to express $\bar{Z}(\cdot)$ as a time change of $\bar{T}'(\cdot)$ and to express (2.3) as a time change of a renewal equation.

Lemma 2.1 *Let $\xi \in \mathcal{M}_F^{c,p}$. Then, for each $t \geq 0$,*

$$\bar{Z}(t) = \left(H'_\xi * U_e\right)(\bar{S}(t)), \tag{2.8}$$

and, for each $t \geq 0$ and $x \in \mathbb{R}_+$,

$$\left\langle 1_{[0,x]}, \bar{\mu}_\xi(t)\right\rangle = \left\langle 1_{(\bar{S}(t),\bar{S}(t)+x]}, \xi\right\rangle + \left(\left(G^x * H_\xi\right) * U_e\right)(\bar{S}(t)). \tag{2.9}$$

Proof. To verify (2.8), use the fact that $\bar{S}(\cdot) = \bar{T}^{-1}(\cdot)$ together with (2.2) to obtain, for each $t \geq 0$,

$$\bar{Z}(t) = \frac{1}{\bar{S}'(t)} = \bar{T}'(\bar{S}(t)).$$

This together with (2.7) implies (2.8). To verify (2.9), use the change of variables $y = \bar{S}(s)$ and the fact that $\bar{S}(\cdot) = \bar{T}^{-1}(\cdot)$ to obtain the following: for each $t \geq 0$ and $x \in \mathbb{R}_+$,

$$\int_0^t G^x(\bar{S}(t) - \bar{S}(s))ds = \int_0^{\bar{S}(t)} G^x(\bar{S}(t) - y)d\bar{T}(y) = \left(G^x * \bar{T}\right)(\bar{S}(t)). \tag{2.10}$$

Substituting (2.10) into (2.3), then using (2.6), and associativity of the convolution operation completes the proof. ■

In the next lemma, we show that, under appropriate conditions, $\bar{S}(t)$ is bounded below by a linear function for all t sufficiently large.

Lemma 2.2 *Given $\eta > 0$, there exists a finite, positive time $T^{\nu,\eta}$ depending on η and ν such that, for all $\xi \in \mathcal{M}_F^{c,p}$ with $\langle \chi, \xi\rangle < \infty$,*

$$\bar{S}(t) \geq \frac{t}{(\beta_e + \eta)\langle \chi, \xi\rangle}, \qquad \textit{for all } t \geq \langle \chi, \xi\rangle\, T^{\nu,\eta}. \tag{2.11}$$

Proof. Fix $\eta > 0$ and $\xi \in \mathcal{M}_F^{c,p}$ such that $\langle \chi, \xi\rangle < \infty$. By the elementary renewal theorem, $U_e(t)/t$ converges to β_e as t tends to infinity (cf. [12], Theorem 3.3.3). We note that this holds even if $\beta_e = 0$, i.e., if $\langle \chi, \nu_e\rangle = \infty$. Thus, there exists a finite, positive time $\tilde{T}^{\nu,\eta}$ such that

$$U_e(t) \leq (\beta_e + \eta)t, \qquad \text{for all } t \geq \tilde{T}^{\nu,\eta}.$$

Note that $\tilde{T}^{\nu,\eta}$ does not depend on ξ since $U_e(\cdot)$ does not depend on ξ. Moreover, since both H_ξ and U_e are nondecreasing, from (2.6) and (2.4), it follows that $\bar{T}(t) \leq H_\xi(t)U_e(t) \leq \langle \chi, \xi \rangle U_e(t)$, for all $t \geq 0$. Thus,

$$\bar{T}(t) \leq (\beta_e + \eta) \langle \chi, \xi \rangle t, \qquad \text{for all } t \geq \tilde{T}^{\nu,\eta}. \tag{2.12}$$

Since $\bar{S}(\cdot) = \bar{T}^{-1}(\cdot)$, it follows that

$$\bar{S}(t) \geq \frac{t}{(\beta_e + \eta) \langle \chi, \xi \rangle}, \qquad \text{for all } t \geq \bar{T}(\tilde{T}^{\nu,\eta}).$$

By (2.12), $\bar{T}(\tilde{T}^{\nu,\eta}) \leq (\beta_e + \eta) \langle \chi, \xi \rangle \tilde{T}^{\nu,\eta}$. Setting $T^{\nu,\eta} = (\beta_e + \eta)\tilde{T}^{\nu,\eta}$, completes the proof. ■

When $\beta_e > 0$, we can set $\eta = \beta_e$ in Lemma 2.2 to obtain the following corollary.

Corollary 2.3 *If $\langle \chi^2, \nu \rangle < \infty$, then there exists a positive, finite time T^ν such that, for all $\xi \in \mathcal{M}_F^{c,p}$ with $\langle \chi, \xi \rangle < \infty$,*

$$\bar{S}(t) \geq \frac{t}{2\beta_e \langle \chi, \xi \rangle}, \qquad \textit{for all } t \geq \langle \chi, \xi \rangle T^\nu. \tag{2.13}$$

3 The Invariant Manifold

Theorem 1.1 is proved in this section. For this, we begin with the following proposition.

Proposition 3.1 *For each $g \in \mathcal{C}$,*

$$\alpha \langle g, \nu \rangle = \langle g', \nu_e \rangle. \tag{3.1}$$

Proof. Fix $g \in \mathcal{C}$. Note that (3.1) may be rewritten as

$$\alpha \int_{\mathbb{R}_+} g(x)dF(x) = \int_{\mathbb{R}_+} g'(x)f_e(x)dx. \tag{3.2}$$

Recall that for real valued, right continuous functions $U(\cdot)$ and $V(\cdot)$ on $\mathbb{R}_+$, that are locally of bounded variation, and such that at least one of U or V is continuous, we have the following integration by parts formula: for all $0 \leq a < b < \infty$,

$$\int_{(a,b]} V(x)dU(x) + \int_{(a,b]} U(x)dV(x) = V(b)U(b) - V(a)U(a), \tag{3.3}$$

(see [7], Theorem 3.30, for example). To prove (3.2), use $F(0) = 0$, (3.3), $g(0) = 0$, $g'(0) = 0$, and g is bounded, together with $\lim_{y\to\infty} F(y) = 1$, to obtain

$$\begin{aligned}\int_{\mathbb{R}_+} g(x)dF(x) &= \int_{(0,\infty)} g(x)dF(x) = \lim_{y\to\infty} \int_{(0,y]} g(x)dF(x) = -\lim_{y\to\infty} \int_{(0,y]} g(x)d\left(1 - F(x)\right) \\ &= -\lim_{y\to\infty} \left[g(y)\left(1 - F(y)\right) - \int_{(0,y]} g'(x)(1 - F(x))dx \right] = \int_{\mathbb{R}_+} g'(x)(1 - F(x))dx.\end{aligned}$$

Since $f_e(x) = \alpha(1 - F(x))$ for all $x \in \mathbb{R}_+$, (3.2) holds, and hence so does (3.1). ∎

Proof of Theorem 1.1. Recall that $\bar{\mu}_{\mathbf{0}}(\cdot) \equiv \mathbf{0}$. Thus, $\xi = \mathbf{0}$ is an invariant state. Therefore, in order to prove Theorem 1.1, it suffices to show that $\xi \in \mathcal{M}_F^{c,p}$ is an invariant state if and only if $\xi = c\nu_e$ for some $c \in (0, \infty)$. Suppose that $\xi \in \mathcal{M}_F^{c,p}$ is an invariant state, i.e., that $\bar{\mu}_\xi(t) \equiv \xi$ for all $t \geq 0$. Fix $t > 0$. Then, since $\bar{\mu}_\xi(\cdot)$ is a fluid model solution, it follows from (1.2) that, for any function $g \in \mathcal{C}$,

$$\int_0^t \frac{\langle g', \bar{\mu}_\xi(s)\rangle}{\langle 1, \bar{\mu}_\xi(s)\rangle} ds = \alpha t \langle g, \nu\rangle . \tag{3.4}$$

Since $\bar{\mu}_\xi(\cdot) \equiv \xi$, it follows that, for each $0 \leq s \leq t$, the numerator and the denominator of the integrand in (3.4) are given by $\langle g', \xi\rangle$ and $\langle 1, \xi\rangle$, respectively. Therefore, (3.4) simplifies to

$$\langle g', \xi\rangle = \langle 1, \xi\rangle \alpha \langle g, \nu\rangle .$$

By (3.1), the right side of the above expression is given by $\langle 1, \xi\rangle \langle g', \nu_e\rangle$. Thus,

$$\langle g', \xi\rangle = c \left\langle g', \nu_e\right\rangle , \qquad \text{for all } g \in \mathcal{C}, \tag{3.5}$$

where $c = \langle 1, \xi\rangle$. It turns out that $\mathcal{C}$ is a sufficiently rich class of functions in order for (3.5) to imply that $\xi = c\nu_e$, where $c = \langle 1, \xi\rangle$. To see this, fix $x \in (0, \infty)$. For $\epsilon \in (0, x/2)$, let $g_\epsilon \in \mathbf{C}_b^1(\mathbb{R}_+)$ such that $0 \leq g_\epsilon' \leq 1$,

$$g_\epsilon'(y) = \begin{cases} 1 & \text{if } y \in (\epsilon, x - \epsilon), \\ 0 & \text{if } y \in [0, \epsilon/2] \cup [x - \epsilon/2, \infty), \end{cases}$$

and $g_\epsilon(y) = \int_0^y g_\epsilon'(z)dz$. Then $g_\epsilon \in \mathcal{C}$. Therefore, by (3.5), $\langle g_\epsilon', \xi\rangle = c\langle g_\epsilon', \nu_e\rangle$, where $c = \langle 1, \xi\rangle$. By letting ϵ tend to zero, it follows from bounded convergence that $\langle 1_{(0,x)}, \xi\rangle = c\left\langle 1_{(0,x)}, \nu_e\right\rangle$, for all $x \in (0, \infty)$. Since neither ξ nor ν_e has an atom at the origin, $\xi = c\nu_e$, where $c = \langle 1, \xi\rangle$. This completes the proof of the "only if" part of the theorem.

For the proof of the "if" part of the theorem, we must show that, if $\xi = c\nu_e$ for some $c \in (0, \infty)$, then ξ is an invariant state, i.e., that $\bar{\mu}_\xi(\cdot) \equiv \xi$. For this let, $\bar{\mu}(\cdot) \equiv \xi$, where $\xi = c\nu_e$ for some $c \in (0, \infty)$. It suffices to show that $\bar{\mu}(\cdot)$ satisfies (1.2). Obviously, for each $g \in \mathcal{C}$,

$$\langle g, \bar{\mu}(t)\rangle = \langle g, \xi\rangle = \langle g, \bar{\mu}(0)\rangle , \qquad \text{for all } t \geq 0. \tag{3.6}$$

By definition of $\bar{\mu}(\cdot)$ and (3.1), for each $g \in \mathcal{C}$,

$$\int_0^t \frac{\langle g', \bar{\mu}(s)\rangle}{\langle 1, \bar{\mu}(s)\rangle} ds = t \left\langle g', \nu_e\right\rangle = \alpha t \langle g, \nu\rangle , \qquad \text{for all } t \geq 0. \tag{3.7}$$

By combining equations (3.6) and (3.7), we see that $\bar{\mu}(\cdot)$ satisfies (1.2), as desired. ∎

4 Weak Convergence to the Invariant Manifold

Theorem 1.2 is proved in this section. The proof given here is an application of the key renewal theorem. Recall that the key renewal theorem implies that for any Borel measurable function $g : \mathbb{R}_+ \to \mathbb{R}_+$ that is directly Riemann integrable, we have

$$\lim_{z\to\infty} \left(g * U_e\right)(z) = \beta_e \int_0^\infty g(x)dx, \tag{4.1}$$

(cf. [6], Chapter XI).

In order to apply the key renewal theorem, we will need to verify that certain functions are directly Riemann integrable. We begin by recalling the definition of the later and some related facts. For $g : \mathbb{R}_+ \to \mathbb{R}_+$ and $n, k \in \{1, 2, \dots\}$, let

$$m_k^n(g) = \inf\{g(z) : z \in [(k-1)/n, k/n)\} \qquad \text{and} \qquad M_k^n(g) = \sup\{g(z) : z \in [(k-1)/n, k/n)\},$$

and define

$$L_n(g) = \frac{1}{n}\sum_{k=1}^\infty m_k^n(g) \qquad \text{and} \qquad U_n(g) = \frac{1}{n}\sum_{k=1}^\infty M_k^n(g).$$

Set

$$\underline{\sigma}(g) = \limsup_{n\to\infty} L_n(g) \qquad \text{and} \qquad \bar{\sigma}(g) = \liminf_{n\to\infty} U_n(g).$$

Clearly, $\underline{\sigma}(g) \le \bar{\sigma}(g)$ since $L_n(g) \le U_n(g)$ for each $n \in \{1, 2, \dots\}$. The function g is said to be directly Riemann integrable if $\bar{\sigma}(g) < \infty$ and $\underline{\sigma}(g) = \bar{\sigma}(g)$. The important point that distinguishes the direct Riemann integral from the traditional Riemann integral is that the sums extend over subintervals covering the entire nonnegative half line, and that, for each n, the supremums and infimums defining m_k^n and M_k^n for $k \in \{1, 2, \dots\}$ are taken over intervals of a fixed length (not varying with k). If g is directly Riemann integrable, then g is Riemann integrable and $\bar{\sigma}(g) = \int_0^\infty g(x)dx$. The converse however is not true. To see this, one only needs to consider a nonnegative, continuous, unbounded, Riemann integrable function.

We now review some sufficient conditions for a function $g : \mathbb{R}_+ \to \mathbb{R}_+$ to be directly Riemann integrable. If g is Riemann integrable on $[0, x]$ for all $x \in \mathbb{R}_+$ and $U_n(g) < \infty$ for some $n \in \{1, 2, \dots\}$, then g is directly Riemann integrable (cf. [6]). In particular, if g is a nonincreasing, Riemann integrable function, then g is directly Riemann integrable since for all $n \in \{1, 2, \dots\}$

$$U_n(g) \le \int_0^\infty g(x)dx + \frac{1}{n}g(0) < \infty.$$

Also, if $g_1 : \mathbb{R}_+ \to \mathbb{R}_+$ is Riemann integrable on $[0, x]$ for all $x \in \mathbb{R}_+$ and satisfies $g_1 \leq g_2$ for some directly Riemann integrable function $g_2 : \mathbb{R}_+ \to \mathbb{R}_+$, then g_1 is also directly Riemann integrable.

Theorem 4.1 *Let $\xi \in \mathcal{M}_F^{c,p}$. If $\langle \chi, \xi \rangle < \infty$, then, for each $x \in \mathbb{R}_+$,*

$$\lim_{t\to\infty} \langle 1_{[0,x]}, \bar{\mu}_\xi(t) \rangle = \beta_e \langle \chi, \xi \rangle \langle 1_{[0,x]}, \nu_e \rangle \qquad \text{and} \qquad \lim_{t\to\infty} \bar{Z}(t) = \beta_e \langle \chi, \xi \rangle .$$

Proof. Fix $\xi \in \mathcal{M}_F^{c,p}$ such that $\langle \chi, \xi \rangle < \infty$. By Lemma 2.2, $\lim_{t\to\infty} \bar{S}(t) = \infty$. Also, the continuous function $H'_\xi(\cdot)$ is directly Riemann integrable since $H'_\xi(\cdot)$ is nonincreasing and

$$\int_0^\infty H'_\xi(x)dx = \langle \chi, \xi \rangle < \infty.$$

These two facts together with (2.8) and (4.1) immediately imply the stated convergence result for $\bar{Z}(\cdot)$.

It remains to prove the stated convergence result for the mass on $[0, x]$, for each $x \in \mathbb{R}_+$. For this, fix $x \in \mathbb{R}_+$. Since the total mass of ξ is finite and since, by Lemma 2.2, $\lim_{t\to\infty} \bar{S}(t) = \infty$, we have

$$\limsup_{t\to\infty} \left\langle 1_{(\bar{S}(t),\bar{S}(t)+x]}, \xi \right\rangle \leq \lim_{t\to\infty} \left\langle 1_{(\bar{S}(t),\infty)}, \xi \right\rangle = 0. \tag{4.2}$$

Thus, the first term on the right side of (2.9) tends to zero as t tends to infinity. To see that the second term on the right side of (2.9) converges to the desired limit, we appeal to the key renewal theorem. For this, let $f_e^x(y) = f_e(x + y)$, for all $y \in \mathbb{R}_+$, and consider the function $(f_e^x * H_\xi)(\cdot)$. Using the fact that H_ξ is continuously differentiable, it can be shown that $(f_e^x * H_\xi)(\cdot)$ is continuous, and therefore it is Riemann integrable over $[0, y]$ for each $y \in \mathbb{R}_+$. Thus, in order to show that $(f_e^x * H_\xi)(\cdot)$ is directly Riemann integrable, it suffices to show that it is bounded above by a function that is directly Riemann integrable. Using the fact that both $f_e^x(\cdot)$ and $H'_\xi(\cdot)$ are nonincreasing, we obtain the following: for each $y \in \mathbb{R}_+$,

$$\begin{aligned}
(f_e^x * H_\xi)(y) &= \int_0^{y/2} f_e^x(y-z)H'_\xi(z)dz + \int_{y/2}^y f_e^x(y-z)H'_\xi(z)dz \\
&\leq f_e^x(y/2) \int_0^{y/2} H'_\xi(z)dz + H'_\xi(y/2) \int_{y/2}^y f_e^x(y-z)dz \\
&\leq f_e^x(y/2) \int_0^\infty H'_\xi(z)dz + H'_\xi(y/2)\left(F_e(x+y/2) - F_e(x)\right) \\
&\leq f_e^x(y/2) \langle \chi, \xi \rangle + H'_\xi(y/2).
\end{aligned}$$

As noted above, $H'_\xi(\cdot)$ is directly Riemann integrable. Since $f_e^x(\cdot)$ is nonincreasing and Riemann integrable, $f_e^x(\cdot)$ is also directly Riemann integrable. Therefore, $(f_e^x * H_\xi)(\cdot)$ is bounded above by a function that is directly Riemann integrable, and hence $(f_e^x * H_\xi)(\cdot)$ is itself directly Riemann integrable. In particular, $(f_e^x * H_\xi)(\cdot)$ is Riemann integrable over $\mathbb{R}_+$ and

$$\begin{aligned}\int_0^\infty \left(f_e^x * H_\xi\right)(y)dy &= \int_0^\infty \int_0^y f_e^x(y-z)H'_\xi(z)dzdy = \int_0^\infty \int_z^\infty f_e^x(y-z)dyH'_\xi(z)dz \\ &= \int_0^\infty \int_0^\infty f_e^x(y)dyH'_\xi(z)dz = \langle \chi, \xi\rangle \int_0^\infty f_e^x(y)dy = \langle \chi, \xi\rangle \int_x^\infty f_e(y)dy.\end{aligned} \tag{4.3}$$

Since $\left(G^x * H_\xi\right)(\cdot) = \left(f_e^0 * H_\xi\right)(\cdot) - \left(f_e^x * H_\xi\right)(\cdot)$, it immediately follows that $\left(G^x * H_\xi\right)(\cdot)$ is directly Riemann integrable. Moreover, by (4.3),

$$\int_0^\infty \left(G^x * H_\xi\right)(y)dy = \langle \chi, \xi\rangle \int_0^x f_e(y)dy = \langle \chi, \xi\rangle F_e(x). \tag{4.4}$$

This together with the key renewal theorem gives

$$\lim_{z\to\infty} \left(\left(G^x * H_\xi\right) * U_e\right)(z) = \beta_e \langle \chi, \xi\rangle F_e(x).$$

Since, by Lemma 2.2, $\lim_{t\to\infty} \bar{S}(t) = \infty$, it follows that

$$\lim_{t\to\infty} \left(\left(G^x * H_\xi\right) * U_e\right)(\bar{S}(t)) = \beta_e \langle \chi, \xi\rangle F_e(x). \tag{4.5}$$

Combining (2.9), (4.2), and (4.5) completes the proof. ∎

Proof of Theorem 1.2. If $\xi = \mathbf{0}$, we have $\bar{\mu}_\xi(\cdot) \equiv \mathbf{0}$, $\beta_e \langle \chi, \xi\rangle = 0$, and the conclusion of Theorem 1.2 holds. Now take $\xi \in \mathcal{M}_F^{c,p}$ such that $\langle \chi, \xi\rangle < \infty$. First suppose that $\left\langle \chi^2, \nu\right\rangle = \infty$. Then $\beta_e = 0$, and by Theorem 4.1, $\bar{Z}(t) \longrightarrow 0$ as $t \to \infty$. Given a continuous, bounded function $g : \mathbb{R}_+ \to \mathbb{R}$, we have, for each $t \geq 0$,

$$|\langle g, \bar{\mu}_\xi(t)\rangle| \leq \sup_{x\in\mathbb{R}_+} |g(x)|\bar{Z}(t).$$

So, it follows that $\bar{\mu}_\xi(t)$ converges weakly to the zero measure as $t \to \infty$. Next suppose that $\left\langle \chi^2, \nu\right\rangle < \infty$, i.e., that $\beta_e > 0$. Recall that $\xi \neq \mathbf{0}$ implies that $\bar{Z}(t) > 0$ for all $t \geq 0$. Therefore, for each $t \geq 0$, one can divide $\bar{\mu}_\xi(t)$ by the total mass to form a probability measure. This normalization of $\bar{\mu}_\xi(t)$ for $t \geq 0$ facilitates the use of standard results on convergence in distribution. For this, for each $t \geq 0$, define the probability distribution function

$$F(t, x) = \frac{\left\langle 1_{[0,x]}, \bar{\mu}_\xi(t)\right\rangle}{\bar{Z}(t)}, \qquad \text{for all } x \in \mathbb{R}_+.$$

By Theorem 4.1, for each $x \in \mathbb{R}_+$, $F(t,x) \longrightarrow F_e(x)$ as $t \to \infty$. It follows that, for any bounded, continuous function $g : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$,

$$\lim_{t\to\infty} \int_{\mathbb{R}_+} g(x) d_x F(t,x) = \int_{\mathbb{R}_+} g(x) dF_e(x),$$

(cf. [4], Chapter 2, Theorem 2.2), i.e., that

$$\lim_{t\to\infty} \frac{\langle g, \bar{\mu}_\xi(t)\rangle}{\bar{Z}(t)} = \langle g, \nu_e \rangle.$$

Since $\lim_{t\to\infty} \bar{Z}(t) = \beta_e \langle \chi, \xi \rangle$, it follows that, for any bounded, continuous function $g : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$,

$$\lim_{t\to\infty} \langle g, \bar{\mu}_\xi(t)\rangle = \beta_e \langle \chi, \xi \rangle \langle g, \nu_e \rangle,$$

which completes the proof. ■

5 A Rate of Convergence in the Prohorov Metric

In this section, we prove Theorem 1.3(i), and in the next section, we prove Theorem 1.3(ii). For this, note that the conditions of part (i) (as well as those of part (ii)) imply that ν satisfies $\langle \chi^2, \nu \rangle < \infty$, and hence $\beta_e > 0$. To ease the typography, for $\beta_e > 0$ and $\xi \in \mathcal{M}_F^c$ satisfying $\langle \chi, \xi \rangle < \infty$, we use the notation

$$\kappa = \beta_e \langle \chi, \xi \rangle . \tag{5.1}$$

We begin with Lemma 5.1 below, which identifies conditions that imply the result of Theorem 1.3(i). For this, recall the definitions given by (1.3) and (1.5).

Lemma 5.1 *Let $M, \epsilon > 0$. Suppose that $\langle \chi^2, \nu \rangle < \infty$ and that there exists a finite constant $C \geq 1$ and a finite time $T \geq 1$ such that for all $\xi \in \mathcal{B}_\rho^{M,\epsilon}$, $t \geq T$, and $0 < x \leq \infty$,*

$$| \langle 1_{[0,x)}, \bar{\mu}_\xi(t) \rangle - \langle 1_{[0,x)}, \kappa\nu_e \rangle | \leq C t^{-\epsilon}. \tag{5.2}$$

Then, for all $\xi \in \mathcal{B}_\rho^{M,\epsilon}$ and $t \geq T$,

$$\rho(\bar{\mu}_\xi(t), \kappa\nu_e) \leq C_\rho t^{-\epsilon/4},$$

where C_ρ is the unique positive root of the polynomial $p(y) = y^2 - (M + 4C)\, y - 2C$, $y \in \mathbb{R}_+$.

Proof. Let $M, \epsilon > 0$ and fix $\xi \in \mathcal{B}_\rho^{M,\epsilon}$. In order to prove Lemma 5.1, it suffices to show that, for each $t \geq T$ and for all nonempty closed sets $B \subset \mathbb{R}_+$,

$$\langle 1_B, \bar{\mu}_\xi(t) \rangle \leq \langle 1_{B^{\delta_t}}, \kappa\nu_e \rangle + \delta_t \qquad \text{and} \qquad \langle 1_B, \kappa\nu_e \rangle \leq \langle 1_{B^{\delta_t}}, \bar{\mu}_\xi(t) \rangle + \delta_t, \tag{5.3}$$

where $\delta_t = C_\rho t^{-\epsilon/4}$. In order to verify (5.3), we begin with a simple observation. As an immediate consequence of (5.2), the fact that $\bar{\mu}_\xi(t)$ has no atoms for each $t \geq 0$, and the fact that ν_e has no atoms, it follows that, for each $0 \leq x < y \leq \infty$,

$$\left| \left\langle 1_{(x,y)}, \bar{\mu}_\xi(t) \right\rangle - \left\langle 1_{(x,y)}, \kappa\nu_e \right\rangle \right| \leq 2Ct^{-\epsilon}, \qquad \text{for all } t \geq T. \tag{5.4}$$

We will use (5.4) in conjunction with (5.2) to verify (5.3). For this, fix a nonempty closed set $B \subset \mathbb{R}_+$ and a finite time $t \geq T$. Note that, since $B \subset B^{\delta_t}$,

$$\langle 1_B, \bar{\mu}_\xi(t) \rangle \leq \langle 1_{B^{\delta_t}}, \bar{\mu}_\xi(t) \rangle \qquad \text{and} \qquad \langle 1_B, \kappa\nu_e \rangle \leq \langle 1_{B^{\delta_t}}, \kappa\nu_e \rangle. \tag{5.5}$$

In order to use (5.2) and (5.4), we will need to write B^{δ_t} as a union of intervals that are relatively open in $\mathbb{R}_+$. Since B^{δ_t} is relatively open in $\mathbb{R}_+$, it is a either a finite or a countable union of relatively open, disjoint intervals. Moreover, by the definition of B^{δ_t}, the length of each interval is at least δ_t. Let N denote the number of these intervals that have nonempty intersection with $[0, t^{\epsilon/2})$. Then, $N \leq \lceil t^{\epsilon/2}/\delta_t \rceil$, where, for all $x \in \mathbb{R}$, $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. Moreover, we can write

$$B^{\delta_t} = I_1 \cup I_2 \cup \cdots \cup I_N \cup \left(B^{\delta_t} \cap (t^{\epsilon/2}, \infty) \right), \tag{5.6}$$

where I_i, $i = 1, \ldots, N$, are relatively open, disjoint intervals in $\mathbb{R}_+$ such that $I_i \cap [0, t^{\epsilon/2}) \neq \emptyset$ for $i = 1, \ldots, N$. Note that, for each $i = 1, \ldots, N$, either $I_i = [0, x)$ for some $0 < x \leq \infty$, or $I_i = (x, y)$ for some $0 \leq x < y \leq \infty$. Using (5.5) and (5.6), together with inequalities (5.2) and (5.4), we have

$$\begin{aligned}
\langle 1_B, \bar{\mu}_\xi(t) \rangle &\leq \sum_{i=1}^N \langle 1_{I_i}, \bar{\mu}_\xi(t) \rangle + \left\langle 1_{(t^{\epsilon/2},\infty)}, \bar{\mu}_\xi(t) \right\rangle \\
&\leq \sum_{i=1}^N \langle 1_{I_i}, \kappa\nu_e \rangle + \left\langle 1_{(t^{\epsilon/2},\infty)}, \kappa\nu_e \right\rangle + (N+1)2Ct^{-\epsilon} \\
&\leq \langle 1_{B^{\delta_t}}, \kappa\nu_e \rangle + \left\langle 1_{(t^{\epsilon/2},\infty)}, \kappa\nu_e \right\rangle + (N+1)2Ct^{-\epsilon}.
\end{aligned}$$

Similarly,

$$\begin{aligned}\langle 1_B, \kappa\nu_e\rangle &\leq \sum_{i=1}^{N} \langle 1_{I_i}, \kappa\nu_e\rangle + \left\langle 1_{(t^{\epsilon/2},\infty)}, \kappa\nu_e\right\rangle \\ &\leq \sum_{i=1}^{N} \langle 1_{I_i}, \bar{\mu}_\xi(t)\rangle + \left\langle 1_{(t^{\epsilon/2},\infty)}, \kappa\nu_e\right\rangle + N2Ct^{-\epsilon} \\ &\leq \langle 1_{B^{\delta_t}}, \bar{\mu}_\xi(t)\rangle + \left\langle 1_{(t^{\epsilon/2},\infty)}, \kappa\nu_e\right\rangle + N2Ct^{-\epsilon}.\end{aligned}$$

Since $\langle \chi^2, \nu\rangle < \infty$, it follows that $\langle \chi, \nu_e\rangle < \infty$. Thus, by (5.1), $\langle \chi, \kappa\nu_e\rangle = \langle \chi, \xi\rangle$. Therefore, $\left\langle 1_{(t^{\epsilon/2},\infty)}, \kappa\nu_e\right\rangle \leq t^{-\epsilon/2} \langle \chi, \kappa\nu_e\rangle = \langle \chi, \xi\rangle\, t^{-\epsilon/2}$. This, together with the fact that $N \leq \left(t^{\epsilon/2}/\delta_t\right) + 1$, gives

$$\langle 1_B, \bar{\mu}_\xi(t)\rangle \leq \langle 1_{B^{\delta_t}}, \kappa\nu_e\rangle + \langle \chi, \xi\rangle\, t^{-\epsilon/2} + \frac{2Ct^{-\epsilon/2}}{\delta_t} + 4Ct^{-\epsilon},$$

and

$$\langle 1_B, \kappa\nu_e\rangle \leq \langle 1_{B^{\delta_t}}, \bar{\mu}_\xi(t)\rangle + \langle \chi, \xi\rangle\, t^{-\epsilon/2} + \frac{2Ct^{-\epsilon/2}}{\delta_t} + 2Ct^{-\epsilon}.$$

Thus, in order to prove (5.3), it suffices to show that

$$\langle \chi, \xi\rangle\, t^{-\epsilon/2} + 4Ct^{-\epsilon} + \frac{2Ct^{-\epsilon/4}}{C_\rho} \leq C_\rho t^{-\epsilon/4},$$

or equivalently, that

$$\left(\langle \chi, \xi\rangle\, t^{-\epsilon/4} + 4Ct^{-3\epsilon/4}\right) C_\rho + 2C \leq C_\rho^2. \tag{5.7}$$

Since $t \geq T \geq 1$ and $\xi \in \mathcal{B}_\rho^{M,\epsilon}$,

$$\langle \chi, \xi\rangle\, t^{-\epsilon/4} + 4Ct^{-3\epsilon/4} \leq \langle \chi, \xi\rangle + 4C \leq M + 4C.$$

Moreover, since C_ρ is a root of $p(\cdot)$, it follows that $(M + 4C)\, C_\rho + 2C = C_\rho^2$. Therefore, (5.7) holds, which implies that (5.3) holds. ■

The next objective is to verify that the sufficient conditions in Lemma 5.1 hold under the conditions in part (i) of Theorem 1.3. It suffices to consider $\xi \neq \mathbf{0}$ since the left side of (5.2) is zero when $\xi = \mathbf{0}$. Note that, given $M, \epsilon > 0$ and $\xi \in \mathcal{B}_\rho^{M,\epsilon}$, by (2.9) and the fact that $\bar{\mu}_\xi(t)$ has no atoms for each $t \geq 0$, we have a representation for the first of the two terms that appear on the left side of (5.2). Also, notice that, for each $t > 0$ and $x \in \mathbb{R}_+$,

$$\left\langle 1_{(\bar{S}(t),\bar{S}(t)+x]}, \xi\right\rangle \leq \left\langle 1_{(\bar{S}(t),\infty)}, \xi\right\rangle \leq \left(\frac{1}{\bar{S}(t)}\right)^{1+\epsilon} \left\langle \chi^{1+\epsilon}, \xi\right\rangle. \tag{5.8}$$

By Corollary 2.3, under appropriate conditions, $\bar{S}(t)$ is bounded below by a linear function for t sufficiently large t (with uniform control for $\xi \in \mathcal{B}_\rho^{M,\epsilon}$). Thus, the most significant issue is to look at the difference between the last term in (2.9) and $\langle 1_{[0,x)}, \kappa\nu_e \rangle$, for each $x \in \mathbb{R}_+$. In the proof of Theorem 1.2, we used the key renewal theorem to show that, under appropriate conditions, for each $x \in \mathbb{R}_+$, the last term in (2.9) converges to $\kappa F_e(x) = \beta_e \int_0^\infty (G^x * H_\xi)(y)dy$ as t tends to infinity (cf. (5.1), (4.4) and (4.5)). Thus, in order to verify that the sufficient conditions in Lemma 5.1 hold, we first identify conditions that yield a rate for this convergence that is uniform over $x \in \mathbb{R}_+$ and $\xi \in \mathcal{B}_\rho^{M,\epsilon}$.

Theorem 5.2 *Let $M, \epsilon > 0$. Suppose that $\langle \chi^{2+\epsilon}, \nu \rangle < \infty$ and that $R : [0,\infty) \longrightarrow \mathbb{R}_+$ is a nonincreasing function for which there exists a finite time $T \geq 2$ and a finite constant $C \geq 1$ such that*

$$|U_e(t+s) - U_e(t) - \beta_e s| \leq CR(t), \qquad \textit{for all } s \in [0,1],\ t \geq T. \tag{5.9}$$

Then there exists a finite constant $\hat{C} \geq 1$ such that, for all $\xi \in \mathcal{B}_\rho^{M,\epsilon}$ and $x \in \mathbb{R}_+$,

$$\left| \beta_e \int_0^\infty \left(G^x * H_\xi\right)(y)dy - \left(\left(G^x * H_\xi\right) * U_e\right)(t) \right| \leq \hat{C}\left(t^{-\epsilon} + R(t/2)\right), \qquad \textit{for all } t \geq T. \tag{5.10}$$

Before proceeding with the proof of Theorem 5.2, we show how to use it in conjunction with Lemma 5.1 to prove part (i) of Theorem 1.3.

Proof of Part (i) of Theorem 1.3. Fix $M, \epsilon > 0$. Suppose that $\langle \chi^{2+\epsilon}, \nu \rangle < \infty$. Let

$$R(t) = \begin{cases} 1, & 0 \leq t < 1, \\ t^{-\epsilon}, & t \geq 1. \end{cases}$$

For each $t \geq 0$, let $D(t,s) = U_e(t+s) - U_e(t) - \beta_e s$, for all $s \geq 0$, and denote by $TV_1(D(t,\cdot))$ the total variation of the function $D(t,\cdot)$ over the interval $[0,1]$. Since, for each $t \geq 0$, $D(t,0) = 0$, it follows that, for each $t \geq 0$,

$$|U_e(t+s) - U_e(t) - \beta_e s| \leq TV_1(D(t,\cdot)), \qquad \text{for all } s \in [0,1]. \tag{5.11}$$

Using coupling techniques, it is possible to obtain bounds on $TV_1(D(t,\cdot))$ for t sufficiently large (cf. [11]). For this, note that, since $\langle \chi^{2+\epsilon}, \nu \rangle < \infty$, it follows that $\langle \chi^{1+\epsilon}, \nu_e \rangle < \infty$. Therefore, equation (6.7)(ii) in Chapter III.6 of [11] with $G \equiv 1$, $U(\cdot) = U_e(\cdot)$, $B = 1$, and $\lambda = \beta_e$ implies that there exists a finite constant $C \geq 1$ and a finite time $T \geq 2$ such that

$$TV_1(D(t,\cdot)) \leq Ct^{-\epsilon}, \qquad \text{for all } t \geq T.$$

This, together with (5.11), implies that (5.9) holds. Thus, by Theorem 5.2, there exists a finite constant $\tilde{C} \geq 1$ (given by $(1+2^\epsilon)$ times the constant $\hat{C}$ in (5.10)) such that for all $\xi \in \mathcal{B}_\rho^{M,\epsilon}$,

$$\left| \beta_e \int_0^\infty \left(G^x * H_\xi\right)(y)dy - \left(\left(G^x * H_\xi\right) * U_e\right)(t) \right| \leq \tilde{C} t^{-\epsilon}, \qquad \text{for all } t \geq T \text{ and } x \in \mathbb{R}_+. \tag{5.12}$$

Fix $\xi \in \mathcal{B}_\rho^{M,\epsilon}$. Using (5.12), together with (2.9), the fact that $\bar{\mu}_\xi(t)$ has no atoms for each $t \geq 0$, the fact that ν_e has no atoms, (4.4), (5.8), and the fact that $T \geq 2$, we obtain, for each $x \in \mathbb{R}_+$ and $t \geq 0$ such that $\bar{S}(t) \geq T$,

$$\begin{aligned} \left|\left\langle 1_{[0,x)}, \bar{\mu}_\xi(t)\right\rangle - \left\langle 1_{[0,x)}, \kappa\nu_e\right\rangle\right| &\leq \left(\frac{1}{\bar{S}(t)}\right)^{1+\epsilon} \left\langle \chi^{1+\epsilon}, \xi\right\rangle + \tilde{C}\left(\frac{1}{\bar{S}(t)}\right)^{\epsilon} \\ &\leq \left(\left\langle \chi^{1+\epsilon}, \xi\right\rangle + \tilde{C}\right)\left(\frac{1}{\bar{S}(t)}\right)^{\epsilon} \leq \left(M + \tilde{C}\right)\left(\frac{1}{\bar{S}(t)}\right)^{\epsilon}. \end{aligned} \tag{5.13}$$

By Corollary 2.3 and the fact that $\xi \in \mathcal{B}_\rho^{M,\epsilon}$, there exists a finite, positive time T^ν (that does not depend on ξ) such that

$$\bar{S}(t) \geq \frac{t}{2\beta_e M}, \qquad \text{for all } t \geq MT^\nu. \tag{5.14}$$

Let $\tilde{T} = \max\{2\beta_e T, T^\nu\}$. Then, for all $t \geq M\tilde{T}$, (5.14) holds and $\bar{S}(t) \geq T$. This together with (5.13) gives, for all $x \in \mathbb{R}_+$ and $t \geq M\tilde{T}$,

$$\left|\left\langle 1_{[0,x)}, \bar{\mu}_\xi(t)\right\rangle - \left\langle 1_{[0,x)}, \kappa\nu_e\right\rangle\right| \leq \left(M + \tilde{C}\right)\left(2\beta_e M\right)^\epsilon t^{-\epsilon}. \tag{5.15}$$

By letting $x \to \infty$ in (5.15), and using the facts that, for each $t \geq 0$, $\bar{\mu}_\xi(t) \in \mathcal{M}_F$ and $\nu_e \in \mathcal{M}_F$, we see that (5.15) holds for $x = \infty$, for all $t \geq M\tilde{T}$. Therefore, (5.2) in Lemma 5.1 holds for the finite constant given by $\left(M + \tilde{C}\right)(2\beta_e M)^\epsilon$ and the finite time given by $M\tilde{T}$. So, Theorem 1.3(i) follows from Lemma 5.1. ■

The final task for this section is to prove Theorem 5.2. For this, we first establish some basic properties of the functions $(G^x * H_\xi)(\cdot)$, for $x \in \mathbb{R}_+$.

Proposition 5.3 *Let $\xi \in \mathcal{M}_F^{c,p}$. For each $x \in \mathbb{R}_+$, the following hold.*

(i) For all $u \geq 0$, $0 \leq \left(G^x * H_\xi\right)(u) \leq \left(f_e * H_\xi\right)(u)$.

(ii) The function $(G^x * H_\xi)(\cdot)$ is absolutely continuous. In particular, for each $u \geq 0$,

$$\left(G^x * H_\xi\right)(u) = \int_0^u L_\xi^x(w)dw, \tag{5.16}$$

where, for each $u \geq 0$,

$$L_\xi^x(u) = G^x(u)H_\xi'(0) - \int_0^u G^x(u-v)\xi(dv). \tag{5.17}$$

(iii) The function $(G^x * H_\xi)(\cdot)$ is of bounded variation. In particular,

$$\int_0^\infty \left|L_\xi^x(u)\right| du \le 3\langle 1, \xi\rangle .$$

(iv) If, for some $\epsilon > 0$, $\langle \chi^{2+\epsilon}, \nu\rangle < \infty$ and $\langle \chi^{1+\epsilon}, \xi\rangle < \infty$, then, for each $x \in \mathbb{R}_+$, the following holds: for all $u > 0$,

$$(G^x * H_\xi)(u) \le \int_u^\infty \left|L_\xi^x(v)\right| dv \le K_\xi u^{-1-\epsilon}, \tag{5.18}$$

where $K_\xi = \left(2^{1+\epsilon}+1\right)\left(\langle \chi^{1+\epsilon}, \nu_e\rangle \langle 1, \xi\rangle + \langle \chi^{1+\epsilon}, \xi\rangle\right)$.

Proof. Fix $\xi \in \mathcal{M}_F^{c,p}$ and $x \in \mathbb{R}_+$. Property (i) is immediate since $0 \le G^x(y) \le f_e(y)$ for all $y \ge 0$ and H_ξ is nondecreasing.

To verify (5.16), note that, by Fubini's theorem, for each $u \ge 0$,

$$\int_0^u \int_0^w G^x(w-v)\xi(dv)dw = \int_0^u \int_v^u G^x(w-v)dw\,\xi(dv) = \int_0^u \int_0^{u-v} G^x(w)dw\,\xi(dv).$$

Recall that, for $y \ge 0$, $H'_\xi(y) = \langle 1_{(y,\infty)}, \xi\rangle$. Thus, $dH'_\xi(v) = -\xi(dv)$. So we have

$$-\int_0^u \int_0^w G^x(w-v)\xi(dv)dw = \int_0^u \int_0^{u-v} G^x(w)dw\, dH'_\xi(v).$$

Thus, regarding $\int_0^{u-v} G^x(w)dw$ as a function of $v \in [0,u]$ and using the integration by parts formula (3.3), we obtain

$$-\int_0^u \int_0^w G^x(w-v)\xi(dv)dw = -H'_\xi(0)\int_0^u G^x(w)dw + \int_0^u G^x(u-v)H'_\xi(v)dv.$$

Then (5.16) follows.

To prove (iii) and (iv), note that, by (ii) and the fact that $G^x(y) \le f_e(y)$ for all $y \ge 0$, for all $u \ge 0$ we have

$$\begin{aligned}\int_u^\infty \left|L_\xi^x(w)\right| dw &\le \int_u^\infty G^x(w)H'_\xi(0)dw + \int_u^\infty \int_0^w G^x(w-v)\xi(dv)dw \\ &\le \int_u^\infty f_e(w)H'_\xi(0)dw + \int_u^\infty \int_0^w f_e(w-v)\xi(dv)dw.\end{aligned}$$

By interchanging the order of integration in the second term on the right side, we obtain, for $u \ge 0$,

$$\begin{aligned}\int_u^\infty \left|L_\xi^x(w)\right| dw &\le H'_\xi(0)\left(1-F_e(u)\right) + \int_0^\infty \int_{v\vee u}^\infty f_e(w-v)dw\,\xi(dv) \\ &\le H'_\xi(0)\left(1-F_e(u)\right) + \int_0^{u/2} \left(1-F_e(u-v)\right)\xi(dv) + \int_{u/2}^\infty \xi(dv).\end{aligned}$$

Now use the fact that $1 - F_e(\cdot)$ is nonincreasing to obtain, for $u \geq 0$,

$$\int_u^\infty \left|L_\xi^x(w)\right| dw \leq H'_\xi(0)\,(1 - F_e(u)) + (1 - F_e(u/2))\, H'_\xi(0) + H'_\xi(u/2). \tag{5.19}$$

To verify (iii), take $u = 0$ in (5.19). To prove (iv), let $\epsilon > 0$ and assume that $\langle \chi^{2+\epsilon}, \nu \rangle < \infty$ and $\langle \chi^{1+\epsilon}, \xi \rangle < \infty$. Then we have $\langle \chi^{1+\epsilon}, \nu_e \rangle < \infty$, and the second inequality in (iv) follows from (5.19) and the fact that for all $t > 0$,

$$H'_\xi(t) \leq \langle \chi^{1+\epsilon}, \xi \rangle\, t^{-1-\epsilon} \qquad \text{and} \qquad 1 - F_e(t) \leq \langle \chi^{1+\epsilon}, \nu_e \rangle\, t^{-1-\epsilon}. \tag{5.20}$$

To prove the first inequality in (iv), note that, since $f_e(\cdot)$ and $H'_\xi(\cdot)$ are nonincreasing, it follows that, for $u > 0$,

$$(f_e * H_\xi)\,(u) \leq H'_\xi(0) \int_{u/2}^\infty f_e(y)dy + f_e(0) \int_{u/2}^\infty H'_\xi(y)dy,$$

where each term on the right side of this inequality is finite for all $u \geq 0$. Therefore, by monotone convergence, each term on the right side of this inequality tends to zero as $u \to \infty$. This together with (i) implies that $\lim_{u\to\infty}(G^x * H_\xi)(u) = 0$. Consequently, by (ii), it follows that

$$(G^x * H_\xi)\,(u) = -\int_u^\infty L_\xi^x(v)dv,$$

from which the first inequality in (iv) follows. ■

To begin the proof of Theorem 5.2, we borrow an idea from the proof of Theorem 3.1 in [10]. The idea is to write $((G^x * H_\xi) * U_e)\,(t)$, for $x \in \mathbb{R}_+$ and $t \geq 0$, as a sum of integrals over intervals of length one (cf. (5.24) below), and then to use integration by parts on each such integral (cf. (5.25) below). In this way, one obtains expressions involving the quantities that appear in (5.9). In the next proposition, a generic term in such a sum is rewritten using integration by parts. Following that, we give the proof of Theorem 5.2.

Proposition 5.4 *Let $x \in \mathbb{R}_+$ and $\xi \in \mathcal{M}_F^{c,p}$. For $t \geq 1$ and $0 \leq s \leq t-1$,*

$$\int_{(s,s+1]} (G^x * H_\xi)\,(t-y)dU_e(y) = (G^x * H_\xi)\,(t-s)\,(U_e(s+1) - U_e(s)) - \int_{[t-s-1,t-s)} (U_e(s+1) - U_e(t-y))\, L_\xi^x(y)dy. \tag{5.21}$$

Proof. Fix $x \in \mathbb{R}_+$ and $t \geq 1$ and $0 \leq s \leq t-1$. Using (5.16) and (3.3), followed by a change of variables, gives

$$\int_{(s,s+1]} (G^x * H_\xi)(t-y)dU_e(y) = (G^x * H_\xi)(t-s-1)U_e(s+1) - (G^x * H_\xi)(t-s)U_e(s) + \int_{[t-s-1,t-s)} U_e(t-y)L_\xi^x(y)dy.$$

Adding and subtracting the term $(G^x * H_\xi)(t-s)U_e(s+1)$ gives

$$\begin{aligned}\int_{(s,s+1]} (G^x * H_\xi)(t-y)dU_e(y) &= (G^x * H_\xi)(t-s)\left(U_e(s+1) - U_e(s)\right) \\ &+ \left((G^x * H_\xi)(t-s-1) - (G^x * H_\xi)(t-s)\right)U_e(s+1) \\ &+ \int_{[t-s-1,t-s)} U_e(t-y)L_\xi^x(y)dy.\end{aligned}$$

Using the fact that

$$(G^x * H_\xi)(t-s-1) - (G^x * H_\xi)(t-s) = -\int_{[t-s-1,t-s)} L_\xi^x(y)dy,$$

and combining like terms, gives the result. ■

Proof of Theorem 5.2. Fix $\xi \in \mathcal{B}_\rho^{M,\epsilon}$ and $x \in \mathbb{R}_+$. For $t \geq T$, let $N_t = \lfloor t - T \rfloor$. For $t \geq T$, we have

$$\left((G^x * H_\xi) * U_e\right)(t) = \int_{[0,t]} (G^x * H_\xi)(t-y)dU_e(y) = I_1(t) + I_2(t), \tag{5.22}$$

where, for $t \geq T$,

$$I_1(t) = \int_{[0,t-N_t]} (G^x * H_\xi)(t-y)dU_e(y) \quad \text{and} \quad I_2(t) = \int_{(t-N_t,t]} (G^x * H_\xi)(t-y)dU_e(y).$$

By parts (i) and (iv) of Proposition 5.3, we have, for $t \geq T$,

$$I_1(t) \leq K_\xi \int_{[0,t-N_t]} (t-y)^{-1-\epsilon}dU_e(y) \leq K_\xi N_t^{-1-\epsilon}U_e(t-N_t).$$

Note that, for $t \geq T$, we have $t - N_t \leq T+1$. In addition, for $t \geq 2T+2$, we have $N_t \geq t/2$. Thus, for $t \geq 2T+2$,

$$I_1(t) \leq 2^{1+\epsilon}K_\xi U_e(T+1)t^{-1-\epsilon}. \tag{5.23}$$

For $I_2(\cdot)$, we have

$$I_2(t) = \sum_{i=1}^{N_t} \int_{(t-N_t+i-1,t-N_t+i]} (G^x * H_\xi)(t-y)dU_e(y), \qquad \text{for all } t \geq T. \tag{5.24}$$

Thus, we can use (5.21), and then the change of variables $j = N_t - i + 1$, to obtain, for $t \geq T$,

$$
\begin{aligned}
I_2(t) &= \sum_{i=1}^{N_t} \left(G^x * H_\xi\right)(N_t - i + 1)\left(U_e(t - N_t + i) - U_e(t - N_t + i - 1)\right) \\
&\quad - \sum_{i=1}^{N_t} \int_{[N_t - i, N_t - i + 1)} \left(U_e(t - N_t + i)) - U_e(t - y)\right) L^x_\xi(y)dy \\
&= \sum_{j=1}^{N_t} \left(G^x * H_\xi\right)(j)\left(U_e(t + 1 - j) - U_e(t - j)\right) \\
&\quad - \sum_{j=1}^{N_t} \int_{[j-1, j)} \left(U_e(t + 1 - j) - U_e(t - y)\right) L^x_\xi(y)dy. \qquad (5.25)
\end{aligned}
$$

Since, for each summand $j \leq N_t$, and since $t - N_t \geq T$, it follows that $t - j \geq T$. Thus, we can use (5.9) on each term in the first sum. Similarly, since in the integrand of each term in the second sum we have $y \leq N_t$, we can use (5.9) on each of these integrands. For $t \geq T$, this gives

$$
|I_2(t) - I_{21}(t)| \leq I_{22}(t) + I_{23}(t), \qquad t \geq T, \qquad (5.26)
$$

where, for $t \geq T$,

$$
\begin{aligned}
I_{21}(t) &= \beta_e \sum_{j=1}^{N_t} \left(\left(G^x * H_\xi\right)(j) - \int_{[j-1,j)} (y + 1 - j) L^x_\xi(y)dy \right), \\
I_{22}(t) &= C \sum_{j=1}^{N_t} \left(G^x * H_\xi\right)(j) R(t - j), \\
I_{23}(t) &= C \int_{[0,N_t)} R(t - y) \left|L^x_\xi(y)\right| dy.
\end{aligned}
$$

The above representation of $I_{21}(\cdot)$ can be simplified. In fact, for $t \geq T$,

$$
I_{21}(t) = \beta_e \int_{[0,N_t)} \left(G^x * H_\xi\right)(z)dz. \qquad (5.27)
$$

To see this, write $y + 1 - j$ as $\int_{j-1}^{y} dz$ and then interchange the order of integration to obtain the following: for all $t \geq T$,

$$
\begin{aligned}
I_{21}(t) &= \beta_e \sum_{j=1}^{N_t} \left((G^x * H_\xi)(j) - \int_{[j-1,j)} \int_{j-1}^{y} dz\ L_\xi^x(y) dy \right) \\
&= \beta_e \sum_{j=1}^{N_t} \left((G^x * H_\xi)(j) - \int_{[j-1,j)} \int_{[z,j)} L_\xi^x(y) dy\ dz \right) \\
&= \beta_e \sum_{j=1}^{N_t} \left((G^x * H_\xi)(j) - \int_{[j-1,j)} \left((G^x * H_\xi)(j) - (G^x * H_\xi)(z) \right)\ dz \right) \\
&= \beta_e \sum_{j=1}^{N_t} \int_{[j-1,j)} (G^x * H_\xi)(z) dz = \beta_e \int_{[0,N_t)} (G^x * H_\xi)(z) dz.
\end{aligned}
$$

Let's now summarize what has been shown. By (5.22), (5.26), and (5.27), for $t \geq T$,

$$
\begin{aligned}
&\left| \beta_e \int_0^\infty (G^x * H_\xi)(z) dz - ((G^x * H_\xi) * U_e))(t) \right| \\
&\qquad \leq \beta_e \int_{[N_t,\infty)} (G^x * H_\xi)(y) dy + I_{22}(t) + I_{23}(t) + I_1(t).
\end{aligned} \tag{5.28}
$$

We have already derived an upper bound on $I_1(\cdot)$ (cf. (5.23)). Next, we obtain estimates on the remaining terms on the right side of the above inequality.

By part (iv) of Proposition 5.3 and the fact that, for $t \geq 2T + 2$, $N_t \geq t/2$, we have, for $t \geq 2T + 2$,

$$
\int_{N_t}^\infty (G^x * H_\xi)(y) dy \leq K_\xi \int_{N_t}^\infty y^{-1-\epsilon} dy = \frac{K_\xi}{\epsilon} N_t^{-\epsilon} \leq 2^\epsilon \frac{K_\xi}{\epsilon} t^{-\epsilon} \tag{5.29}
$$

To obtain a bound on $I_{22}(\cdot)$, note that, for $t \geq T$,

$$
\begin{aligned}
&\sum_{j=1}^{N_t} R(t-j)\,(G^x * H_\xi)(j) \\
&\qquad = \sum_{j \in [1,t/2)} R(t-j)\,(G^x * H_\xi)(j) + \sum_{j \in [t/2,N_t]} R(t-j)\,(G^x * H_\xi)(j).
\end{aligned}
$$

By part (iv) of Proposition 5.3, the fact that $R(\cdot)$ is nonincreasing, and the fact that $t - N_t \geq T$ for $t \geq T$, we obtain, for $t \geq T$,

$$
\begin{aligned}
\sum_{j \in [t/2,N_t]} R(t-j)\,(G^x * H_\xi)(j) &\leq R(t - N_t) K_\xi \sum_{j \in [t/2,N_t]} j^{-1-\epsilon} \\
&\leq R(T) K_\xi \int_{t/2-1}^\infty y^{-1-\epsilon} dy \\
&\leq \frac{R(T) K_\xi}{\epsilon} \left(\frac{t}{2} - 1 \right)^{-\epsilon}.
\end{aligned}
$$

Note that, for $t \geq 2T+2$, we have $1 - 2/t \geq T/(T+1)$, and so

$$\left(\frac{t}{2} - 1\right)^{-\epsilon} \leq \left(\frac{T+1}{T}\right)^{\epsilon} \left(\frac{t}{2}\right)^{-\epsilon}.$$

Thus, for $t \geq 2T+2$,

$$\sum_{j \in [t/2, N_t]} R(t-j) \left(G^x * H_\xi\right)(j) \leq \frac{R(T) K_\xi 2^\epsilon}{\epsilon} \left(\frac{T+1}{T}\right)^{\epsilon} t^{-\epsilon}.$$

Since $R(\cdot)$ is nonincreasing, it also follows from part (iv) of Proposition 5.3 that, for $t \geq 2T+2$,

$$\sum_{j \in [1, t/2)} R(t-j) \left(G^x * H_\xi\right)(j) \leq R(t/2) \sum_{j \in [1, t/2)} \left(G^x * H_\xi\right)(j) \leq K_\xi R(t/2) \sum_{j \in [1, \infty)} j^{-1-\epsilon}.$$

Thus, we have, for $t \geq 2T+2$,

$$I_{22}(t) \leq C \frac{R(T) K_\xi 2^\epsilon}{\epsilon} \left(\frac{T+1}{T}\right)^{\epsilon} t^{-\epsilon} + C K_\xi \left(\sum_{j=1}^{\infty} j^{-1-\epsilon}\right) R(t/2). \tag{5.30}$$

We now bound $I_{23}(\cdot)$. In a similar manner to that used above, for $t \geq 2T+2$, write

$$\begin{aligned} I_{23}(t) &= C \int_{[0, N_t)} R(t-y) \left|L_\xi^x(y)\right| dy \\ &= C \int_{[0, t/2)} R(t-y) \left|L_\xi^x(y)\right| dy + C \int_{[t/2, N_t)} R(t-y) \left|L_\xi^x(y)\right| dy. \end{aligned}$$

Then, since $R(\cdot)$ is nonincreasing, it follows from parts (iii) and (iv) of Proposition 5.3 that, for $t \geq 2T+2$,

$$\begin{aligned} I_{23}(t) &\leq C R(t/2) \int_0^\infty \left|L_\xi^x(y)\right| dy + C R(0) \int_{[t/2, \infty)} \left|L_\xi^x(y)\right| dy \\ &\leq 3C \langle 1, \xi \rangle R(t/2) + 2^{1+\epsilon} C R(0) K_\xi t^{-1-\epsilon}. \end{aligned} \tag{5.31}$$

Combining (5.28), (5.29), (5.30), (5.31), and (5.23) with the fact that, since $\xi \in \mathcal{B}_\rho^{M,\epsilon}$,

$$K_\xi \leq M \left(2^{1+\epsilon} + 1\right) \left(\left\langle \chi^{1+\epsilon}, \nu_e \right\rangle + 1\right),$$

proves the desired result. ■

6 A Rate of Convergence in the Total Variation Distance

Theorem 1.3(ii) is proved in this section. For this, fix $M, \epsilon > 0$. Throughout this section, we assume that

$$\left\langle \chi^{3+\epsilon}, \nu \right\rangle < \infty. \tag{6.1}$$

Note that, for $\xi = \mathbf{0}$, (1.8) holds for any positive constant C_{TV} and any positive time T_{TV}. To see this, observe that, if $\xi = \mathbf{0}$, then $\kappa = 0$ (cf. (5.1)). Moreover, $\bar{\mu}_{\mathbf{0}}(\cdot) \equiv \mathbf{0}$, and therefore, for all $t \geq 0$, $\|\bar{\mu}_\xi(t) - \kappa\nu_e\|_{\mathrm{TV}} = 0$ if $\xi = \mathbf{0}$. So it suffices to prove (1.8) for $\xi \in \mathcal{B}_{\mathrm{TV}}^{M,\epsilon}$ such that $\xi \neq \mathbf{0}$.

Fix $\xi \in \mathcal{B}_{\mathrm{TV}}^{M,\epsilon}$ and $\xi \neq \mathbf{0}$. The first order of business is to obtain an upper bound on $\|\bar{\mu}_\xi(t) - \kappa\nu_e\|_{\mathrm{TV}}$, for $t \geq 0$, that is comprised of three terms (cf. (6.6) below). Then a rate of convergence to zero as t tends to infinity is obtained for each of the three terms. For this, we need to introduce some notation. For a function $g : \mathbb{R}_+ \to \mathbb{R}$ that is locally of bounded variation, let $TV_x(g)$ denote the total variation of g on $[0, x]$, for each $x \in \mathbb{R}_+$. Also, denote the total variation of g by $TV(g) = \lim_{x\to\infty} TV_x(g)$. Let

$$J(t,x) = \left\langle 1_{[0,x]}, \bar{\mu}_\xi(t) \right\rangle - \kappa \left\langle 1_{[0,x]}, \nu_e \right\rangle, \qquad \text{for all } t \geq 0,\ x \in \mathbb{R}_+.$$

Note that, for each $t \geq 0$, neither $\bar{\mu}_\xi(t)$ nor ν_e charges the origin. Therefore, for all $t \geq 0$,

$$\|\bar{\mu}_\xi(t) - \kappa\nu_e\|_{\mathrm{TV}} = TV(J(t,\cdot)). \tag{6.2}$$

For each $t \geq 0$, the function $J(t,\cdot)$ is readily expressed as three distinct terms. To see this, note that, by (2.3), (2.10), and the definition of $F_e(\cdot)$, it follows that, for all $t \geq 0$ and $x \in \mathbb{R}_+$,

$$J(t,x) = \left\langle 1_{(\bar{S}(t),\bar{S}(t)+x]}, \xi \right\rangle + \left(G^x * \bar{T}\right)(\bar{S}(t)) - \kappa F_e(x). \tag{6.3}$$

Clearly, since $\bar{S}(t)$ tends to infinity as t tends to infinity, the total variation of the first term on the right side of (6.3) tends to zero as t tends to infinity. But individually, the total variation of the second and third terms on the right side of (6.3) fail to converge to zero. Therefore, it will be necessary to take advantage of the minus sign. This can be done by expressing the third term on the right side of (6.3) as a sum of two terms (cf. Lemma 6.1 below), the first of which combines with the second term on the right side of (6.3) to form a term whose total variation tends to zero as t tends to infinity. For this, it will be convenient to view the convolution in the second term on the right side of (6.3) as a convolution of a function with a measure. We make the following definition. Given a signed Radon measure ζ and a bounded, Borel measurable function $g : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, let

$$(g * \zeta)(x) = \int_{[0,x]} g(x-y)\zeta(dy), \qquad \text{for all } x \in \mathbb{R}_+.$$

Let τ be the Radon measure on $\mathbb{R}_+$ such that

$$\left\langle 1_{[0,x]}, \tau \right\rangle = \bar{T}(x), \qquad \text{for all } x \in \mathbb{R}_+. \tag{6.4}$$

Then, for all $t \geq 0$ and $x \in \mathbb{R}_+$,

$$J(t,x) = \left\langle 1_{(\bar{S}(t),\bar{S}(t)+x]}, \xi \right\rangle + \left(G^x * \tau\right)\left(\bar{S}(t)\right) - \kappa F_e(x). \tag{6.5}$$

To obtain an upper bound on $TV(J(t,\cdot))$, for each $x \in \mathbb{R}_+$ we will express $F_e(x)$ as a convolution of $G^x(\cdot)$ with Lebesgue measure on $\mathbb{R}_+$, plus a remainder term. For this, let ℓ denote Lebesgue measure on $\mathbb{R}_+$.

Lemma 6.1 *For $x \in \mathbb{R}_+$, $F_e(x) = \left(G^x * \ell\right)\left(\bar{S}(t)\right) + \left\langle 1_{(\bar{S}(t),\bar{S}(t)+x]}, \nu_e \right\rangle$.*

Proof. For each $x \in \mathbb{R}_+$,

$$\begin{aligned} F_e(x) &= \int_0^x f_e(y)dy = \left(\int_0^\infty f_e(y)dy - \int_x^\infty f_e(y)dy\right) \\ &= \left(\int_0^\infty f_e(y)dy - \int_0^\infty f_e(x+y)dy\right) = \int_0^\infty G^x(y)dy. \end{aligned}$$

Splitting this integral into two pieces gives, for $x \in \mathbb{R}_+$ and $t \geq 0$,

$$\begin{aligned} F_e(x) &= \int_0^{\bar{S}(t)} G^x(y)\ell(dy) + \int_{\bar{S}(t)}^\infty G^x(y)\ell(dy) \\ &= \int_0^{\bar{S}(t)} G^x(\bar{S}(t)-y)\ell(dy) + \int_{\bar{S}(t)}^\infty \left(f_e(y) - f_e(x+y)\right)\ell(dy) \\ &= \left(G^x * \ell\right)\left(\bar{S}(t)\right) + \int_{\bar{S}(t)}^{x+\bar{S}(t)} f_e(y)\ell(dy) = \left(G^x * \ell\right)\left(\bar{S}(t)\right) + \left\langle 1_{(\bar{S}(t),\bar{S}(t)+x]}, \nu_e \right\rangle. \end{aligned}$$

■

For each $t \geq 0$ and $x \in (0,\infty)$, let

$$\begin{aligned} A(t,x) &= \left\langle 1_{(\bar{S}(t),\bar{S}(t)+x]}, \xi \right\rangle, \\ B(t,x) &= \left(G^x * \tau\right)\left(\bar{S}(t)\right) - \kappa\left(G^x * \ell\right)\left(\bar{S}(t)\right), \\ C(t,x) &= \kappa\left\langle 1_{(\bar{S}(t),\bar{S}(t)+x]}, \nu_e \right\rangle. \end{aligned}$$

Also, for each $t \geq 0$, set $A(t,0) = 0$, $B(t,0) = 0$, and $C(t,0) = 0$. Note that $A(t,\cdot)$, $B(t,\cdot)$, and $C(t,\cdot)$ are right continuous. Then, by (6.5) and Lemma 6.1, for all $t \geq 0$ and $x \in \mathbb{R}_+$,

$$J(t,x) = A(t,x) + B(t,x) - C(t,x).$$

Thus, by (6.2), for all $t \geq 0$,

$$\left\|\bar{\mu}_\xi(t) - \kappa\nu_e\right\|_{\mathrm{TV}} \leq TV(A(t,\cdot)) + TV(B(t,\cdot)) + TV(C(t,\cdot)). \tag{6.6}$$

In order to prove part (ii) of Theorem 1.3, we will bound each term on the right side of (6.6) from above.

Since, for each $t \geq 0$, the functions $A(t, \cdot)$ and $C(t, \cdot)$ are nondecreasing,

$$TV(A(t,\cdot)) = \lim_{x\to\infty} A(t,x) - A(t,0) = \left\langle 1_{(\bar{S}(t),\infty)}, \xi \right\rangle, \tag{6.7}$$

$$TV(C(t,\cdot)) = \lim_{x\to\infty} C(t,x) - C(t,0) = \kappa \left\langle 1_{(\bar{S}(t),\infty)}, \nu_e \right\rangle. \tag{6.8}$$

By (6.7), (6.8), Chebyshev's inequality, (6.1), (2.13), (5.1), and the fact that $\xi \in \mathcal{B}_{\mathrm{TV}}^{M,\epsilon}$, it follows that for $t \geq MT^\nu$,

$$TV(A(t,\cdot)) \leq (\bar{S}(t))^{-2-\epsilon} \langle \chi^{2+\epsilon}, \xi \rangle \leq (2\beta_e)^{2+\epsilon} M^{3+\epsilon} t^{-2-\epsilon}, \tag{6.9}$$

$$TV(C(t,\cdot)) \leq \kappa (\bar{S}(t))^{-2-\epsilon} \langle \chi^{2+\epsilon}, \nu_e \rangle \leq 2^{2+\epsilon} (\beta_e M)^{3+\epsilon} \langle \chi^{2+\epsilon}, \nu_e \rangle t^{-2-\epsilon}. \tag{6.10}$$

Thus, for a proof of (1.8), we have obtained suitable upper bounds on the first and last terms on the right side of (6.6).

The remaining task is to bound $TV(B(t,\cdot))$ from above, for t sufficiently large. Observe that, for all $t \geq 0$ and $x \in \mathbb{R}_+$,

$$\begin{aligned} B(t,x) &= (G^x * (\tau - \kappa\ell))\,(\bar{S}(t)) = ((f_e - f_e^x) * (\tau - \kappa\ell))\,(\bar{S}(t)) \\ &= \alpha\,((F^x - F) * (\tau - \kappa\ell))\,(\bar{S}(t)) \\ &= \alpha\,(F^x * (\tau - \kappa\ell))\,(\bar{S}(t)) - \alpha\,(F * (\tau - \kappa\ell))\,(\bar{S}(t)), \end{aligned}$$

where, for each $x \in \mathbb{R}_+$, $F^x(y) = F(x+y)$ for all $y \in \mathbb{R}_+$. Note that $(F * (\tau - \kappa\ell))\,(\bar{S}(t))$ does not depend on x. Hence, it makes no contribution to the total variation of $B(t,\cdot)$. Thus, for each $t \geq 0$,

$$TV(B(t,\cdot)) = \alpha\, TV(D(t,\cdot)), \tag{6.11}$$

where $D(t,x) = (F^x * (\tau - \kappa\ell))\,(\bar{S}(t))$ for all $t \geq 0$ and $x \in \mathbb{R}_+$. To obtain a suitable upper bound on the total variation of $D(t,\cdot)$ for all t sufficiently large, we introduce the following additional notation. For a signed Radon measure ζ on $\mathbb{R}_+$, let $|\zeta|$ denote the total variation measure of ζ and let ζ^+ and ζ^- be nonnegative Radon measures such that $\zeta = \zeta^+ - \zeta^-$ and $|\zeta| = \zeta^+ + \zeta^-$.

Lemma 6.2 *Let ζ be a signed Radon measure on $\mathbb{R}_+$. For fixed $r \geq 0$, define two functions $g(x) = (F^x * \zeta)\,(r)$ and $\hat{g}(x) = (F^x * |\zeta|)\,(r)$ for all $x \in \mathbb{R}_+$. Then $TV(g) \leq TV(\hat{g})$.*

Proof. Fix $x \in \mathbb{R}_+$ and $h > 0$. We have

$$g(x+h) - g(x) = \int_0^r \left(F(x+h+r-y) - F(x+r-y)\right)\zeta(dy).$$

Since F is nondecreasing,

$$|g(x+h) - g(x)| \leq \int_0^r \left(F(x+h+r-y) - F(x+r-y)\right)|\zeta|(dy) = |\hat{g}(x+h) - \hat{g}(x)|\,.$$

The result follows from the definition of $TV(\cdot)$. ■

By Lemma 6.2, for each $t \geq 0$,

$$TV(D(t,\cdot)) \leq TV(\hat{D}(t,\cdot)), \tag{6.12}$$

where $\hat{D}(t,x) = (F^x * |\tau - \kappa\ell|)\,(\bar{S}(t))$ for all $t \geq 0$ and $x \in \mathbb{R}_+$. Since $F(\cdot)$ is nondecreasing, $\hat{D}(t,\cdot)$ is also nondecreasing for each fixed $t \geq 0$. Therefore,

$$TV(\hat{D}(t,\cdot)) = \lim_{x\to\infty} \hat{D}(t,x) - \hat{D}(t,0), \qquad \text{for all } t \geq 0. \tag{6.13}$$

By monotone convergence,

$$\lim_{x\to\infty} \hat{D}(t,x) = \int_0^{\bar{S}(t)} |\tau - \kappa\ell|(dy), \qquad \text{for all } t \geq 0.$$

Therefore, by (6.13),

$$TV(\hat{D}(t,\cdot)) = \int_0^{\bar{S}(t)} \left(1 - F(\bar{S}(t) - y)\right)|\tau - \kappa\ell|(dy), \qquad \text{for all } t \geq 0. \tag{6.14}$$

When considering why (6.14) should be small when t is large, one realizes that for large values of the argument y, the measures τ and $\kappa\ell$ are close, while for small values of the argument y, the function $1 - F(\bar{S}(t) - y)$ is small. In order to take advantage of this, fix $\delta \in (0,1)$. Given $t \geq 0$, rewrite the above integral as two pieces:

$$\int_0^{(1-\delta)\bar{S}(t)} \left(1 - F(\bar{S}(t) - y)\right)|\tau - \kappa\ell|\,(dy), \tag{6.15}$$

$$\int_{(1-\delta)\bar{S}(t)}^{\bar{S}(t)} \left(1 - F(\bar{S}(t) - y)\right)|\tau - \kappa\ell|\,(dy). \tag{6.16}$$

We begin by analyzing (6.15). For each $t \geq 0$,

$$\begin{aligned}\int_0^{(1-\delta)\bar{S}(t)} \left(1 - F(\bar{S}(t) - y)\right)|\tau - \kappa\ell|(dy) &\leq \left(1 - F(\delta\bar{S}(t))\right)\int_0^{(1-\delta)\bar{S}(t)} |\tau - \kappa\ell|(dy) \\ &\leq \left(1 - F(\delta\bar{S}(t))\right)\|\tau - \kappa\ell\|_{\mathrm{TV}}\,. \end{aligned}\tag{6.17}$$

Using Chebyshev's inequality, (6.1), (2.13), and the fact that $\xi \in \mathcal{B}_{\mathrm{TV}}^{M,\epsilon}$, gives, for each $t \geq M\,T^{\nu}$,

$$\left(1 - F(\delta \bar{S}(t))\right) \leq \left\langle \chi^{3+\epsilon}, \nu \right\rangle \left(\delta \bar{S}(t)\right)^{-3-\epsilon} \leq \left\langle \chi^{3+\epsilon}, \nu \right\rangle \left(\frac{2\beta_e M}{\delta}\right)^{3+\epsilon} t^{-3-\epsilon}. \tag{6.18}$$

Combining (6.17) and (6.18), we have, for each $t \geq M\,T^{\nu}$,

$$\int_0^{(1-\delta)\bar{S}(t)} \left(1 - F(\bar{S}(t) - y)\right) |\tau - \kappa \ell|(dy) \leq \|\tau - \kappa\ell\|_{\mathrm{TV}} \left\langle \chi^{3+\epsilon}, \nu \right\rangle \left(\frac{2\beta_e M}{\delta}\right)^{3+\epsilon} t^{-3-\epsilon}. \tag{6.19}$$

We now analyze (6.16). For each $t \geq 0$,

$$\int_{(1-\delta)\bar{S}(t)}^{\bar{S}(t)} \left(1 - F(\bar{S}(t) - y)\right) |\tau - \kappa\ell|(dy) \leq \int_{(1-\delta)\bar{S}(t)}^{\bar{S}(t)} |\tau - \kappa\ell|(dy) \leq \left\langle 1_{[\bar{S}(t)(1-\delta),\infty)}, |\tau - \kappa\ell| \right\rangle. \tag{6.20}$$

Then, by combining (6.14), (6.19), and (6.20), we have for each $t \geq M\,T^{\nu}$,

$$TV(\hat{D}(t,\cdot)) \leq \|\tau - \kappa\ell\|_{\mathrm{TV}} \left\langle \chi^{3+\epsilon}, \nu \right\rangle \left(\frac{2\beta_e M}{\delta}\right)^{3+\epsilon} t^{-3-\epsilon} + \left\langle 1_{[\bar{S}(t)(1-\delta),\infty)}, |\tau - \kappa\ell| \right\rangle. \tag{6.21}$$

From (6.21), we see that what is needed are estimates on

$$\|\tau - \kappa\ell\|_{\mathrm{TV}} \qquad \text{and} \qquad \left\langle 1_{[r,\infty)}, |\tau - \kappa\ell| \right\rangle, \tag{6.22}$$

for large r. Recall that $\kappa = \beta_e \langle \chi, \xi \rangle$ and that $\xi \neq \mathbf{0}$. So, after factoring out $\langle \chi, \xi \rangle$ from each of the expressions in (6.22), it suffices to obtain estimates on

$$\|\tau / \langle \chi, \xi \rangle - \beta_e \ell\|_{\mathrm{TV}} \qquad \text{and} \qquad \left\langle 1_{[r,\infty)}, |\tau / \langle \chi, \xi \rangle - \beta_e \ell| \right\rangle,$$

for large r. We note that $\beta_e \ell$ is a stationary renewal measure. To see this, consider a renewal process for which the interarrival distribution is determined by ν_e and the initial delay distribution is determined by $(\nu_e)_e$, where $(\nu_e)_e$ is the excess lifetime probability measure associated with ν_e. Specifically, $(\nu_e)_e$ is the Borel probability measure on $\mathbb{R}_+$ that is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}_+$ and has density function

$$\beta_e \left(1 - F_e(x)\right), \qquad \text{for all } x \in \mathbb{R}_+.$$

Here note that, by (6.1), $\beta_e > 0$. This renewal process is stationary, and for any Borel set $A \subset \mathbb{R}_+$, $\beta_e \langle 1_A, \ell \rangle$ is the expected number of arrivals that occur in the set A (cf. [11], Chapter III.2, (2.1)). Also notice that $H_\xi(\cdot)/\langle \chi, \xi \rangle$ is a probability distribution function on $\mathbb{R}_+$. In fact, it has density function $H'_\xi(\cdot)/\langle \chi, \xi \rangle$, which makes it the excess lifetime distribution function for the Borel probability measure $\xi / \langle 1, \xi \rangle$ on $\mathbb{R}_+$ (cf. (2.5)). Let ξ_e denote the Borel probability measure on

$\mathbb{R}_+$ associated with the distribution function $H_\xi(\cdot)/\langle\chi,\xi\rangle$. The observation that $H_\xi(\cdot)/\langle\chi,\xi\rangle$ is a probability distribution function, together with (6.4) and (2.6), implies that that $\tau/\langle\chi,\xi\rangle$ is the renewal measure associated with the renewal process for which the interarrival distribution is determined by ν_e and the initial delay distribution is determined by ξ_e (cf. [11], Chapter III.1, (1.4)(ii)). Therefore, what is needed are estimates on the rate at which the delayed renewal measure $\tau/\langle\chi,\xi\rangle$ converges to the stationary renewal measure $\beta_e\ell$.

One powerful tool that yields rates of convergence to stationarity for renewal measures is coupling (cf. [11]). In fact, under certain conditions, it is possible to couple two renewal processes with a common interarrival distribution so that the respective excess lifetimes agree forever after some random time ς called the coupling time. In our case, the common interarrival distribution is determined by ν_e and the initial delay distributions are determined by ξ_e and $(\nu_e)_e$ respectively. In addition, the coupling time ς is finite a.s. due to (6.1) and the fact that $\xi\in\mathcal{B}_{\mathrm{TV}}^{M,\epsilon}$ (cf. [11], Chapter III.5). Furthermore, the results in [11] state that if the initial delay distributions and the interarrival distribution have finite γ^{th} moments, then the coupling time ς has a finite γ^{th} moment (cf. [11] Chapter III.6, (6.2)). Thus, by (6.1) and the fact that $\xi\in\mathcal{B}_{\mathrm{TV}}^{M,\epsilon}$, it follows that $\mathbf{E}[\varsigma^\gamma]<\infty$ for all $\gamma\in[0,1+\epsilon]$, where $\mathbf{E}$ denotes expected value. In fact, by carefully following the discussion on pages 83 and 84 in [11], which explains how to adapt of the proof of Theorem II.4.2 in [11] from the discrete time setting to the continuous time setting, and by carefully keeping track of the constants used in that argument, one can verify that, for $\gamma\in[1,1+\epsilon]$,

$$\mathbf{E}[\varsigma^\gamma]\le\frac{6^\gamma\langle\chi^{1+\gamma},\xi\rangle}{(1+\gamma)\langle\chi,\xi\rangle}+\frac{\langle\chi^2,\xi\rangle}{2\langle\chi,\xi\rangle}C_1^\nu(\gamma)+C_2^\nu(\gamma),\tag{6.23}$$

where $C_1^\nu(\gamma)$ and $C_2^\nu(\gamma)$ are finite, positive constants that depend on ν and γ, but do not depend on ξ. In particular, since $\xi\in\mathcal{B}_{\mathrm{TV}}^{M,\epsilon}$,

$$\langle\chi,\xi\rangle\,\mathbf{E}[\varsigma]\le\left(3+\frac{C_1^\nu(1)}{2}+C_2^\nu(1)\right)M,\tag{6.24}$$

$$\langle\chi,\xi\rangle\,\mathbf{E}\left[\varsigma^{1+\epsilon}\right]\le\left(\frac{6^{1+\epsilon}}{2+\epsilon}+\frac{C_1^\nu(1+\epsilon)}{2}+C_2^\nu(1+\epsilon)\right)M.\tag{6.25}$$

Since it is more than a simple exercise to obtain (6.23) from the details included in [11], the verification of (6.23) is included as an appendix here (cf. Appendix A.2).

Next we show how to use (6.24) and (6.25) to obtain bounds on $TV(\hat{D}(t,\cdot))$ for $t\ge0$. For this, recall that by (6.21) it suffices to obtain bounds on the quantities that appear in (6.22). By carefully following the arguments on pages 84 and 85 of [11], it can be shown that for $r \geq 1$,

$$\|\tau - \kappa\ell\|_{\mathrm{TV}} \leq 2\langle\chi,\xi\rangle\, U_e(1)\,(1+\mathbf{E}(\varsigma)) \tag{6.26}$$

$$\left\langle 1_{[r,\infty)}, |\tau-\kappa\ell|\right\rangle \leq 2\langle\chi,\xi\rangle\, U_e(1)\mathbf{E}\left(\varsigma^{1+\epsilon}\right) r^{-\epsilon}, \tag{6.27}$$

(cf. Appendix A.3). Combining (6.26) and (6.27) with (6.21), (2.13), and the fact that $\xi \in \mathcal{B}_{\mathrm{TV}}^{M,\epsilon}$, gives the following bound on $TV(\hat{D}(\cdot,\cdot))$: for all $t \geq M\,T^{\nu}$,

$$TV(\hat{D}(t,\cdot)) \leq 2\langle\chi,\xi\rangle\; U_e(1)\left((1+\mathbf{E}(\varsigma))\left\langle\chi^{3+\epsilon},\nu\right\rangle\left(\frac{2\beta_e M}{\delta}\right)^{3+\epsilon} t^{-3} + \mathbf{E}\left(\varsigma^{1+\epsilon}\right)\left(\frac{2\beta_e M}{1-\delta}\right)^{\epsilon}\right) t^{-\epsilon}. \tag{6.28}$$

Combining (6.28) with (6.24) and (6.25) provides a bound on $TV(\hat{D}(t,\cdot))$ for $t \geq 0$ of the type that is needed to complete the proof of Theorem 1.3(ii).

Proof of Part (ii) of Theorem 1.3. Fix $M,\epsilon > 0$. If $\xi = \mathbf{0}$, it follows that $\|\bar{\mu}_\xi(t) - \kappa\nu_e\|_{\mathrm{TV}} = 0$ for all $t \geq 0$. Therefore, it suffices to show that there exists a finite positive constant C_{TV} and a finite positive time T_{TV} such that (1.8) holds for all $\xi \in \mathcal{B}_{\mathrm{TV}}^{M,\epsilon}$ such that $\xi \neq \mathbf{0}$. This follows from (6.1), (6.6), (6.9), (6.10), (6.11), (6.12), (6.28), (6.24) and (6.25). ■

A Appendix

In this appendix, we verify (6.23), (6.26), and (6.27), which were used in the proof of Theorem 1.3(ii). For this, fix $M,\epsilon > 0$ and $\xi \in \mathcal{B}_{\mathrm{TV}}^{M,\epsilon}$ such that $\xi \neq \mathbf{0}$. Throughout the appendix, it assumed that (6.1) holds.

The proofs of (6.23), (6.26), and (6.27) hinge on using the general coupling construction given in Chapter III.5 of Lindvall [11] to couple two renewal processes with a common interarrival distribution determined by ν_e and initial delay distributions determined by ξ_e and $(\nu_e)_e$, respectively. We refer to such renewal processes as ξ_e-delay and stationary renewal processes, respectively. Given a ξ_e-delay (resp. stationary) renewal process, let $N(\cdot)$ (resp. $N^s(\cdot)$) denote the associated counting measure. Here the superscript s stands for stationary. Thus, for each Borel set $A \subset \mathbb{R}_+$,

$$\mathbf{E}[N(A)] = \frac{\langle 1_A,\tau\rangle}{\langle\chi,\xi\rangle} \qquad \text{and} \qquad \mathbf{E}[N^s(A)] = \beta_e\langle 1_A,\ell\rangle\,, \tag{A.1}$$

where τ is defined by (6.4) and ℓ denotes Lebesgue measure. Also, for $n \in \{1,2,\dots\}$, let T_n (resp. T_n^s) denote the time of the nth arrival in the ξ_e-delay (resp. stationary) renewal process. By convention, set $T_0 = T_0^s = 0$. For $t \geq 0$, let

$$A(t) = \min\{t - T_n \geq 0 : n = 0, 1, 2, \dots\},$$
$$D(t) = \min\{T_n - t > 0 : n = 0, 1, 2, \dots\}.$$

At time t, $A(t)$ is the time that has elapsed since the most recent arrival in the ξ_e-delay renewal process, i.e., the *age* of the most recent arrival. Similarly, $D(t)$ is the time that will elapse beginning from time t until the next arrival in the ξ_e-delay renewal process, i.e., the *delay* until the next arrival. For the stationary renewal process, the age process $A^s(\cdot)$ and the delay process $D^s(\cdot)$ are defined in an analogous fashion. The reason for referring to the renewal process with initial delay distribution determined by $(\nu_e)_e$ as a stationary renewal process is that, for each $t \geq 0$, the distribution of $D^s(t)$ is equal to that of $D^s(0)$, which is determined by $(\nu_e)_e$.

The coupling construction in [11] uses various properties of zero-delay renewal processes, which are renewal processes with initial delay distribution determined by δ_0, where δ_0 is the probability measure that puts one unit of mass at the origin. Given such a renewal process with interarrival distribution determined by ν_e, the associated counting measure and other processes are defined in a manner analogous to that for the ξ_e-delay and stationary renewal processes, except that they are distinguished by the presence of a superscript z ($N^z(\cdot)$, $T_\cdot^z$, $A^z(\cdot)$, and $D^z(\cdot)$). Note that $N^z(\{0\}) = 1$, $T_1^z = 0$ almost surely, and for each $t \geq 0$,

$$\mathbf{E}\left[N^z([0,t])\right] = U_e(t), \tag{A.2}$$

(cf. [11], Chapter III.1, (1.4)(i)). Properties of the distribution of $A^z(t)$, for t sufficiently large, are used in determining the frequency of coupling attempts. Specifically, since ν_e has a density (which implies that it is "spread out"), by Lemma III.5.1 in [11], there exist finite, positive constants m, k, and T such that, for each $t \geq T$, the distribution of $A^z(t)$ has an absolutely continuous component for which the density is bounded below by m on $[0, k]$ and $1 - F_e(k) > 0$. For the remainder of this Appendix, we fix such a triple, (m, k, T). Note that these constants depend only on ν, and not on ξ, since it is zero-delay renewal processes that are under consideration here.

We begin in Section A.1 by summarizing some important properties of the coupling construction given in [11]. Then, in Section A.2, we use these properties to derive a bound that is sufficient to imply (6.23) (cf. Theorem A.1). Finally (6.26) and (6.27) are verified in Section A.3.

A.1 The Coupling Time

For the case where the initial delay distributions are determined by ξ_e and $(\nu_e)_e$, the interarrival distributions are determined by ν_e, and the triple (associated with the interarrival distribution ν_e) is given by (m, k, T), the coupling construction in Chapter III.5 of [11] yields a ξ_e-delay renewal process and a stationary renewal process, both defined on the same probability space, with certain additional properties, some of which we describe below. For this we use the same notation for the interarrival times, age processes and delay processes associated with these two renewal processes as established at the beginning of the Appendix. In addition, we let $W_0 = 0$, $n_0 = 0$, $n_0^s = 0$, and for $i \in \{1, 2, 3, \dots\}$, we iteratively define

$$Z_{i-1} = \max\{D(W_{i-1}), D^s(W_{i-1})\}, \qquad W_i = W_{i-1} + Z_{i-1} + T,$$

$$n_i = \max\{n : T_n \leq W_i\}, \qquad n_i^s = \max\{n : T_n^s \leq W_i\}.$$

Finally, we let

$$\mathcal{T} = \min\{i \geq 1 : D(W_i) = D^s(W_i)\}. \tag{A.3}$$

The coupling construction in [11] is such that

$$\mathbf{P}\left(\mathcal{T} < \infty\right) = 1 \qquad \text{and} \qquad D(t) = D^s(t), \qquad \text{for all } t \geq W_{\mathcal{T}},$$

(cf. [11], page 81). In fact, by (III.5.3) in [11], there exists $\delta \in (0, 1]$, that does not depend on ξ, such that

$$\mathbf{P}(\mathcal{T} \geq i) \leq (1 - \delta)^{i-1}, \qquad \text{for } i = 1, 2, \dots. \tag{A.4}$$

Using the fact that the interarrival distribution is determined by ν_e, it is possible to show that (A.4) holds for $\delta = m^2(1 - F_e(k))k^2$. The *coupling time* ς is given by

$$\varsigma = Z_0 + \sum_{i=1}^{\mathcal{T}} (T + Z_i) = Z_0 + \sum_{i=1}^{\infty} 1_{\{\mathcal{T} \geq i\}}(T + Z_i). \tag{A.5}$$

The times W_i, $i = 1, \dots, \mathcal{T}$, are the times at which coupling attempts were made. For $1 \leq i < \mathcal{T}$, each attempt was unsuccessful since $D(W_i) \neq D^s(W_i)$. However, the $\mathcal{T}$th such attempt was successful since $D(W_{\mathcal{T}}) = D^s(W_{\mathcal{T}})$. The coupling construction is such that between successive coupling attempts, the coupled renewal processes satisfy a conditional independence property, which we now describe. For this, for $i \in \{0, 1, 2, \dots\}$, let

$$\mathcal{F}_i = \sigma\{T_{n \wedge (n_i + 1)}, T^s_{n \wedge (n_i^s + 1)} : n = 0, 1, 2, \dots\}.$$

For fixed $i \in \{1, 2, 3, \dots\}$, conditioning on $\mathcal{F}_i$ allows the two renewal processes to be restarted at the arrival times T_{n_i+1} and $T^s_{n^s_i+1}$, respectively. When the two renewal processes conditioned on $\mathcal{F}_i$ are restarted at their respective renewal arrival times, the coupling construction ensures that, on $\{\mathcal{T} > i\}$, they evolve as independent zero-delay renewal processes for $T + Z_i - D(W_i)$ and $T + Z_i - D^s(W_i)$ units of time, respectively. This conditional independence property is important for the proofs given below.

A.2 Bounds for Moments of the Coupling Time

In this section, we prove the following theorem, which implies (6.23).

Theorem A.1 *Let $\gamma \in [1, 1+\epsilon]$. Then ς has a finite γ^{th} moment. Moreover,*

$$(\mathbf{E}[\varsigma^\gamma])^{1/\gamma} \le (\mathbf{E}[Z_0^\gamma])^{1/\gamma} + \left(2^\gamma T^\gamma + 2^{2\gamma+1} C_1 (T+1) + 2^{2\gamma+1} C_1 \mathbf{E}[Z_0]\right)^{1/\gamma} + \frac{\left(\left(2^\gamma T^\gamma + 2^{2\gamma+1} C_1 (T+1)\right)(1-\delta) + 2^{2\gamma+2} C_1 C_2\right)^{1/\gamma}}{1 - (1-\delta)^{1/\gamma}},$$

where T is as in Section A.1, $\delta \in (0,1]$ is as in (A.4) *and C_1 and C_2 are finite, positive constants that do not depend on ξ (but may depend on ν and γ).*

We begin by showing how to obtain (6.23) from Theorem A.1.

Proof of (6.23). Fix $\gamma \in [1, 1+\epsilon]$. Recall that T and δ do not depend on ξ. Let

$$K_1 = 2^\gamma T^\gamma + 2^{2\gamma+1} C_1 (T+1),$$
$$K_2 = 2^{2\gamma+1} C_1,$$
$$K_3 = \frac{\left(\left(2^\gamma T^\gamma + 2^{2\gamma+1} C_1 (T+1)\right)(1-\delta) + 2^{2\gamma+2} C_1 C_2\right)^{1/\gamma}}{1 - (1-\delta)^{1/\gamma}}.$$

Thus, K_1, K_2, and K_3 do not depend on ξ. By Theorem A.1,

$$\begin{aligned}
\mathbf{E}[\varsigma^\gamma] &\le \left((\mathbf{E}[Z_0^\gamma])^{1/\gamma} + (K_1 + K_2 \mathbf{E}[Z_0])^{1/\gamma} + K_3\right)^\gamma \\
&\le \left(3 \max\left\{(\mathbf{E}[Z_0^\gamma])^{1/\gamma}, (K_1 + K_2 \mathbf{E}[Z_0])^{1/\gamma}, K_3\right\}\right)^\gamma \\
&\le 3^\gamma \mathbf{E}[Z_0^\gamma] + 3^\gamma (K_1 + K_2 \mathbf{E}[Z_0]) + 3^\gamma K_3^\gamma. \qquad (A.6)
\end{aligned}$$

Since $Z_0 \le D(0) + D^s(0)$, it follows that $Z_0^\gamma \le 2^\gamma (D(0))^\gamma + 2^\gamma (D^s(0))^\gamma$. Therefore,

$$\mathbf{E}[Z_0] \le \mathbf{E}[D(0)] + \mathbf{E}[D^s(0)] = \langle \chi, \xi_e \rangle + \langle \chi, (\nu_e)_e \rangle, \qquad (A.7)$$

$$\mathbf{E}[Z_0^\gamma] \le 2^\gamma \mathbf{E}[(D(0))^\gamma] + 2^\gamma \mathbf{E}[(D^s(0))^\gamma] = 2^\gamma \langle \chi^\gamma, \xi_e \rangle + 2^\gamma \langle \chi^\gamma, (\nu_e)_e \rangle. \qquad (A.8)$$

It is easily verified that

$$\langle \chi, \xi_e \rangle = \frac{\langle \chi^2, \xi \rangle}{2\,\langle \chi, \xi \rangle} \qquad \text{and} \qquad \langle \chi^\gamma, \xi_e \rangle = \frac{\langle \chi^{1+\gamma}, \xi \rangle}{(1+\gamma)\,\langle \chi, \xi \rangle}. \tag{A.9}$$

Combining (6.1), (A.6), (A.7), (A.8), and (A.9) proves (6.23). ■

The remaining task is to prove Theorem A.1. For this, we apply some of the general arguments given in [11] to the special case where the interarrival distribution is determined by ν_e and the initial delays are given by $(\nu_e)_e$ and ξ_e respectively. Since [11] does not indicate how the various constants that appear in the proofs depend on the initial delay ξ_e, we provide enough details here to keep track of this dependence. For this, we follow the arguments on pages 83–84 in [11], filling in certain details and carefully keeping track of the constants and what they depend on. These general arguments exploit certain properties of zero-delay renewal processes. For our purposes, the statements in Lemma A.2 below suffice. Note that, in Lemma A.2, it is the zero-delay renewal process that is being considered. Therefore, the constants C_1 and C_2 do not depend on ξ. However, they do depend on ν and the constant C_1 may also depend on γ.

Lemma A.2

(i) For each $\gamma \in [0, 2+\epsilon]$, there exists a finite, positive constant C_1 such that, for all $t \geq 0$, $\mathbf{E}\left[(D^z(t))^\gamma\right] \leq C_1(t+1)$.

(ii) There exists a finite, positive constant C_2 such that $\mathbf{E}\left[D^z(t)\right] \leq C_2$ for all $t \geq 0$.

Proof. Fix $\gamma \in [0, 2+\epsilon]$ and $t \geq 0$. Note that by (6.1), $\langle \chi^\gamma, \nu_e \rangle < \infty$. From page 84 of [11], it follows that

$$\mathbf{E}\left[(D^z(t))^\gamma\right] \leq \langle \chi^\gamma, \nu_e \rangle\, U_e(t).$$

By (6.1), $\langle \chi^2, \nu_e \rangle < \infty$. Therefore, by Lorden's inequality (cf. [11], Chapter III, (4.1) (ii)),

$$\mathbf{E}\left[(D^z(t))^\gamma\right] \leq \frac{\langle \chi^\gamma, \nu_e \rangle}{\langle \chi, \nu_e \rangle} t + \frac{\langle \chi^\gamma, \nu_e \rangle \langle \chi^2, \nu_e \rangle}{\langle \chi, \nu_e \rangle^2},$$

which proves (i). To prove (ii), note that

$$D^z(t) = T^z_{N^z([0,t])+1} - t.$$

Therefore, by (A.2) and Wald's identity,

$$\mathbf{E}\left[D^z(t)\right] = \langle \chi, \nu_e \rangle \left(U_e(t) + 1\right) - t.$$

Then, by Lorden's inequality,

$$\mathbf{E}\left[D^z(t)\right] \leq t + \frac{\langle \chi^2, \nu_e \rangle}{\langle \chi, \nu_e \rangle} + \langle \chi, \nu_e \rangle - t = \langle \chi, \nu_e \rangle + \frac{\langle \chi^2, \nu_e \rangle}{\langle \chi, \nu_e \rangle},$$

which completes the proof. ■

Proof of Theorem A.1. Fix $\gamma \in [1, 1+\epsilon]$. By (A.5) and Minkowski's inequality,

$$(\mathbf{E}\left[\varsigma^\gamma\right])^{1/\gamma} \leq (\mathbf{E}\left[Z_0^\gamma\right])^{1/\gamma} + \sum_{i=1}^{\infty} \left(\mathbf{E}\left[(T + Z_i)^\gamma 1_{\{\mathcal{T} \geq i\}}\right]\right)^{1/\gamma}. \tag{A.10}$$

Since $\gamma \in [1, 1+\epsilon]$, by (6.1), $\langle \chi^\gamma, (\nu_e)_e \rangle < \infty$. Also, since $\xi \in \mathcal{B}_{\mathrm{TV}}^{M,\epsilon}$, $\langle \chi^\gamma, \xi_e \rangle < \infty$. Therefore, $\mathbf{E}\left[Z_0^\gamma\right] < \infty$ (cf. (A.8)). The next objective is to bound from above $\mathbf{E}\left[(T + Z_i)^\gamma 1_{\{\mathcal{T} \geq i\}}\right]$, for each $i \in \{1, 2, 3, \dots\}$. For this, fix $i \in \{1, 2, 3, \dots\}$. Note that $1_{\{\mathcal{T} \geq i\}} \in \mathcal{F}_{i-1}$. Moreover, by using the inequality $(x+y)^\gamma \leq 2^\gamma (x^\gamma + y^\gamma)$, for $x, y \in \mathbb{R}_+$, it follows that,

$$\mathbf{E}\left[(T + Z_i)^\gamma \mid \mathcal{F}_{i-1}\right] \leq 2^\gamma T^\gamma + 2^\gamma \mathbf{E}\left[Z_i^\gamma \mid \mathcal{F}_{i-1}\right].$$

By definition, $Z_i = \max\{D(W_i), D^s(W_i)\}$. Therefore,

$$\mathbf{E}\left[Z_i^\gamma \mid \mathcal{F}_{i-1}\right] \leq 2^\gamma \mathbf{E}\left[(D(W_i))^\gamma \mid \mathcal{F}_{i-1}\right] + 2^\gamma \mathbf{E}\left[(D^s(W_i))^\gamma \mid \mathcal{F}_{i-1}\right].$$

By Lemma A.2(i) and the conditional independence property of the coupling construction, it follows that, on $\{\mathcal{T} \geq i\}$,

$$\begin{aligned}
\mathbf{E}\left[Z_i^\gamma \mid \mathcal{F}_{i-1}\right] &\leq 2^\gamma C_1 (T + Z_{i-1} - D(W_{i-1}) + 1) + 2^\gamma C_1 (T + Z_{i-1} - D^s(W_{i-1}) + 1) \\
&\leq 2^{\gamma+1} C_1 (T + 1 + Z_{i-1}).
\end{aligned}$$

Thus, on $\{\mathcal{T} \geq i\}$,

$$\mathbf{E}\left[(T + Z_i)^\gamma \mid \mathcal{F}_{i-1}\right] \leq 2^\gamma T^\gamma + 2^{2\gamma+1} C_1 (T + 1 + Z_{i-1}).$$

This together with (A.4) implies that,

$$\mathbf{E}\left[(T + Z_i)^\gamma 1_{\{\mathcal{T} \geq i\}}\right] \leq \left(2^\gamma T^\gamma + 2^{2\gamma+1} C_1 (T+1)\right)(1-\delta)^{i-1} + 2^{2\gamma+1} C_1 \mathbf{E}\left[Z_{i-1} 1_{\{\mathcal{T} \geq i\}}\right].$$

If $i = 1$, we obtain

$$\left(\mathbf{E}\left[(T + Z_1)^\gamma 1_{\{\mathcal{T} \geq 1\}}\right]\right)^{1/\gamma} \leq \left(2^\gamma T^\gamma + 2^{2\gamma+1} C_1 (T+1) + 2^{2\gamma+1} C_1 \mathbf{E}\left[Z_0\right]\right)^{1/\gamma}. \tag{A.11}$$

If $i \geq 2$, then making the observation that $1_{\{\mathcal{T} \geq i\}} \leq 1_{\{\mathcal{T} \geq i-1\}}$ and conditioning on $\mathcal{F}_{i-2}$ gives

$$\begin{aligned} \mathbf{E}\left[(T+Z_i)^\gamma 1_{\{\mathcal{T} \geq i\}}\right] \leq \left(2^\gamma T^\gamma + 2^{2\gamma+1} C_1 (T+1)\right)(1-\delta)^{i-1} \\ + 2^{2\gamma+1} C_1 \mathbf{E}\left[\mathbf{E}\left[Z_{i-1} \mid \mathcal{F}_{i-2}\right] 1_{\{\mathcal{T} \geq i-1\}}\right]. \end{aligned}$$

Recall that $Z_{i-1} = \max\{D(W_{i-1}), D^s(W_{i-1})\} \leq D(W_{i-1}) + D^s(W_{i-1})$. Thus, if $i \geq 2$, Lemma A.2(ii) and the conditional independence property of the coupling construction imply that, on $\{\mathcal{T} \geq i-1\}$,

$$\mathbf{E}\left[Z_{i-1} \mid \mathcal{F}_{i-2}\right] \leq 2C_2.$$

Therefore, if $i \geq 2$,

$$\left(\mathbf{E}\left[(T+Z_i)^\gamma 1_{\{\mathcal{T} \geq i\}}\right]\right)^{1/\gamma} \leq \left(\left(2^\gamma T^\gamma + 2^{2\gamma+1} C_1 (T+1)\right)(1-\delta) + 2^{2\gamma+2} C_1 C_2\right)^{1/\gamma} (1-\delta)^{(i-2)/\gamma}. \tag{A.12}$$

Combining (A.10), (A.11), and (A.12) completes the proof. ∎

A.3 Verification of (6.26) and (6.27)

Here, we apply the general arguments given on pages 84-85 in [11] to the particular circumstances of interest here while keeping track of the constants to verify (6.26) and (6.27). For this, note that, for each $t \geq 0$,

$$\left\langle 1_{[t,t+1)}, |\tau - \kappa \ell| \right\rangle = \langle \chi, \xi \rangle \left\langle 1_{[t,t+1)}, |\tilde{\tau} - \beta_e \ell| \right\rangle,$$

where $\tilde{\tau} = \tau / \langle \chi, \xi \rangle$. Recall that $\tilde{\tau}$ and $\beta_e \ell$ are the renewal measures associated with the coupled ξ_e-delay and stationary renewal processes described in Section A.1. As noted in the introduction to this appendix, for each $t \geq 0$, the Borel probability measure corresponding to the distribution of the delay $D^s(t)$ is given by $(\nu_e)_e$, for all $t \geq 0$. For each $t \geq 0$, let $\xi_e(t)$ denote the Borel probability measure corresponding to the distribution of the delay $D(t)$. For each $t \geq 0$, by restarting each process at time t, it follows that

$$\left\langle 1_{[t,t+1)}, |\tilde{\tau} - \beta_e \ell| \right\rangle \leq U_e(1) \left\| \xi_e(t) - (\nu_e)_e \right\|_{\mathrm{TV}}, \tag{A.13}$$

(cf. [11], (III.6.6)). By the coupling-mapping inequality (cf. [11], (I.2.12)), for each $t \geq 0$,

$$\left\| \xi_e(t) - (\nu_e)_e \right\|_{\mathrm{TV}} \leq 2\mathbf{P}(\varsigma > t). \tag{A.14}$$

By (A.13) and (A.14), for each $r \geq 0$,

$$\left\langle 1_{[r,\infty)}, |\tilde{\tau} - \beta_e \ell| \right\rangle \leq \sum_{i=0}^{\infty} \left\langle 1_{[r+i,r+i+1)}, |\tilde{\tau} - \beta_e \ell| \right\rangle \leq U_e(1) \sum_{i=0}^{\infty} 2\mathbf{P}(\varsigma > r+i).$$

In the above inequality, for each $i \in \{0, 1, 2, \dots\}$, replace $\mathbf{P}\left(\varsigma > r+i\right)$ with

$$\sum_{j=i}^{\infty} \mathbf{P}\left(r+j < \varsigma \leq r+j+1\right),$$

interchange the order of summation, and simplify, to obtain, for each $r \geq 0$,

$$\left\langle 1_{[r,\infty)}, |\tilde{\tau} - \beta_e \ell| \right\rangle \leq 2U_e(1) \sum_{j=0}^{\infty} (j+1)\mathbf{P}(r+j < \varsigma \leq r+j+1). \tag{A.15}$$

Letting $r = 0$ in (A.15) gives

$$\|\tilde{\tau} - \beta_e \ell\|_{\mathrm{TV}} \leq 2U_e(1)\left(1 + \mathbf{E}\left[\varsigma\right]\right).$$

Multiplying this by $\langle \chi, \xi \rangle$ proves (6.26). To verify (6.27), fix $r \geq 1$. Then, from (A.15), using the fact that $r \geq 1$, it follows that

$$\left\langle 1_{[r,\infty)}, |\tilde{\tau} - \beta_e \ell| \right\rangle \leq 2U_e(1) \sum_{j=0}^{\infty} (r+j) P(r+j < \varsigma \leq r+j+1).$$

Note that, for each $j \geq 0$, $1 \leq r^{-\epsilon}(r+j)^{\epsilon}$. So it follows that

$$\left\langle 1_{[r,\infty)}, |\tilde{\tau} - \beta_e \ell| \right\rangle \leq 2U_e(1) \left(\sum_{j=0}^{\infty} (r+j)^{1+\epsilon} P(r+j < \varsigma \leq r+j+1) \right) r^{-\epsilon} \leq 2U_e(1)\mathbf{E}\left[\varsigma^{1+\epsilon}\right] r^{-\epsilon}.$$

Multiplying the above inequality by $\langle \chi, \xi \rangle$ proves (6.27). ■

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