Practice Problems for Math 20B

Do not forget to give reasons for your answers. Answers without explanation of how you obtained them will not give you credit in the final, as will sketches which are not labeled!

1) Find the following limits.

a) $\lim_{x\to\infty} xe^{-x}$ b) $\lim_{x\to 1} \sqrt{x-1}\ln(x-1)$

c) $\lim_{x\to 1} \frac{1-x}{1-\sqrt{x}}$ d) $\lim_{x\to 0} x^{\alpha}\ln x, \quad \alpha > 0$

2) Find the quadratic Taylor polynomial $T_2(x)$ approximating $\sqrt{1+x^2}$ at $x = 0$.

3) Let $F(x)$ be defined by the formula

$$F(x) = \int_0^{e^x} \sin(t^2)\,dt.$$

Is F differentiable? Explain why. If F is differentiable, find its derivative.

4) Use properties of integrals to show that

$$\int_0^{\pi/2} x\sin(x^12)\,dx \le \frac{\pi^2}{8}.$$

5) Tom visits his mother, who lives 15 miles away. At 11.45 am his mother calls him and he still has not left. However, he makes it to lunch, which is served at 12.00 pm sharp. Whatever means of transportation Tom used, show that his speed at some time between 11.45 and 12.00 was 60 mph.

6) Paul dives from the 10 meter board into the pool. How long does it take until he hits the water? At which speed does he hit the water? (Approximate the earth's gravity by $9.8m/sec^2$).

7) Find $\int_0^4 |2x-1|\,dx$ by interpreting this integral in terms of areas.

8) The density ρ of a rod of length $10m$ is given by the formula $\rho(x) = 5+\sqrt{|5-x|}$, $0 \le x \le 10$. What is the total mass of the rod?

9) Use induction to show that

$$\left(1+\frac{1}{1}\right)^1\left(1+\frac{1}{2}\right)^2\left(1+\frac{1}{3}\right)^3\cdots\left(1+\frac{1}{n-1}\right)^{n-1} = \frac{n^n}{n!}$$

for all $n \ge 2$.

10) Find the following integrals.

a) $\int_{-1}^{1} e^{-x^2}\sin(x^3)\,dx$ b) $\int \frac{\sin 2x}{\sqrt{1+\cos^2 x}}\,dx$

c) $\int \frac{e^x}{e^x+e^{-x}}\,dx$ d) $\int e^{\sqrt{x}}\,dx$

11 Find the number a such that the line $y = a$ disects the area between $y = 1-x^2$ and $y = -2$.

12 Find the volume of a ball of radius R by rotating a half-circle around the x-axis (or the y-axis, your choice).

13 Prove the reduction formula

$$\int \sec^n(x)\,dx = \frac{\sec^{n-2}(x)\tan(x)}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}(x)\,dx$$

and use it to find $\int \sec^3 x\,dx$. For which values of x are the formulas valid?

14) Find the following integrals:

$$a)\int \sin^6 x \cos^9 x\,dx \qquad b)\int \tan^3 x \sec x\,dx \qquad c)\int \tan^5 x \sec^2 x\,dx$$

15) Find the following integrals:

$$a)\int_0^1 x^5\sqrt{1+x^2}\,dx \qquad b)\int_0^2 x^3\sqrt{4-x^2}\,dx$$

$$c)\int \sqrt{1+x^2}\,dx \qquad d)\int \frac{\sqrt{x^2-4}}{x^4}\,dx$$

16) Find the following integrals:

$$a)\int \frac{e^x+2}{e^{2x}+e^x} \qquad b)\int \frac{dx}{x^{1/3}-x^{1/2}} \qquad c)\int \ln(1+\sqrt{x})\,dx$$

17) If we are using Simpson's rule to approximate the integral $\int_a^b f(x)\,dx$, the error E_S is bounded by

$$|E_S| \leq \frac{K(b-a)^5}{180n^4},$$

where n is the number of subdivisions used and K is a constant such that $K \geq |f^(4)(x)|$ for all x, $a \leq x \leq b$ (here $f^{(4)}$ denotes the fourth derivative of f).
18) Determine whether the following improper integrals converge. If they do, find their value.

$$\text{a)}\int_0^\infty xe^{-x}\,dx \qquad \text{b)}\int_0^1 x^k \ln x\,dx, \quad k=1,2,3,\dots$$

19) Use Euler's Method with stepsize 0.1 to estimate the solution of the initial value problem $y' = x + y$, $y(0) = 1$ at $x = 0.5$.
20) The population of some countries has been shown to follow the differential equation $P' = \alpha P^\beta$, where $\alpha > 0$, $\beta > 1$. Solve this differential equation and show that its solutions "explode" in finite time T. Find this time for a specific initial value $P(0) = P_0$.
21) The population P of *Drosophila*, a certain fruit fly, has been shown to be modeled by

$$\frac{dP}{dt} = \frac{P}{5} - \frac{P^2}{5175}.$$

Solve this differential equation and given an initial population of 120 fruitflies, find the population after 20 days.