

4.8 American Put Option

Consider an American put option with strike price $K \in [0, \infty)$ and expiration time T . Then this has payoff process $Y = \{Y_t, t \in [0, T]\}$, where

$$Y_t = (K - S_t)^+, \quad t \in [0, T].$$

Since Y is a continuous, adapted process bounded by K , and $Y_t \geq 0$ for all $t \in [0, T]$, Y satisfies the hypotheses of Section 4.6. Thus, by Theorem 4.6.2, the initial arbitrage free price for this American put option is given by

$$U_0 = \sup_{\tau \in \mathcal{T}_{[0, T]}} E^{P^*} [e^{-r\tau} (K - S_\tau)^+]. \quad (4.95)$$

The right member above is difficult to compute directly as it involves taking a supremum over an uncountable family of stopping times. In fact, for each $t \in [0, T]$, the value of $U_t = e^{rt} U_t^*$, for U_t^* as defined in (4.79), can be characterized in terms of a solution of a free boundary problem for a partial differential equation. Here we shall motivate and state results associated with this characterization. However, for a detailed development including proofs, we refer the reader to the more advanced text on Mathematical Finance by Karatzas and Shreve [11], Section 2.7.

Recall that under P^* ,

$$S_t = S_0 \exp \left(\sigma \tilde{W}_t - \frac{1}{2} \sigma^2 t + rt \right), \quad t \in [0, T], \quad (4.96)$$

where $\tilde{W}$ is a standard Brownian motion. Now, for each $T \in [0, \infty)$ and $x \in [0, \infty)$, let $v(T, x)$ denote the value of the right member of (4.95) given that $S_0 = x$. Note that $v(0, x) = (K - x)^+$. (Although T has so far been fixed in this chapter and $S_0 > 0$, for the definition of $v(T, x)$, we allow T to vary and allow $S_0 = 0$ as a possibility. This causes no difficulty, in particular, one may simply think of using a different Black-Scholes model for each T in computing the value of $v(T, x)$.)

Proposition 4.8.1 *The function $v : [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$ is continuous, and satisfies $v(t, x) \geq (K - x)^+$ for all $t \in [0, \infty)$ and $x \in [0, \infty)$.*

PROOF. See Proposition 7.1 in Section 2.7 of [11]. $\square$

We now revert to $T \in (0, \infty)$ being fixed. Then,

$$U_0 = v(T, S_0). \quad (4.97)$$

In fact, one can express the entire process $\{U_t, t \in [0, T]\}$, which turns out to be the arbitrage free price process for the American put (see Exercise 7 below),

in terms of the function v , as follows. For this, we note that the filtration $\{\mathcal{F}_t, t \in [0, T]\}$ is the same as that generated by the process $\{S_t, t \in [0, T]\}$ under P^* (this includes augmentation of the filtration by the P^* -null sets). By the stationary independent increments property of $\tilde{W}$ under P^* , $\{S_t, t \in [0, T]\}$ is a strong Markov process. In particular, using this property, it can be shown that for each $t \in [0, T]$,

$$U_t = \text{ess sup}_{\tau \in \mathcal{T}_{[t, T]}} E^{P^*} \left[e^{-r(\tau-t)} (K - S_\tau)^+ \mid \mathcal{F}_t \right] \quad (4.98)$$

$$= v(T - t, S_t). \quad (4.99)$$

Recall the definition of τ^* from Section 4.6 and the fact that $\{U_{t \wedge \tau^*}^*, \mathcal{F}_t, t \in [0, T]\}$ is a martingale under P^* . Using the above we have

$$\tau^* = \inf\{s \geq 0 : v(T - s, S_s) = (K - S_s)^+\}. \quad (4.100)$$

This motivates introducing the continuation region:

$$\mathcal{C} = \{(t, x) \in (0, \infty) \times (0, \infty) : v(t, x) > (K - x)^+\},$$

and the stopping region:

$$\mathcal{D} = \{(t, x) \in [0, \infty) \times [0, \infty) : v(t, x) = (K - x)^+\}.$$

For each $t \in (0, \infty)$, let

$$c(t) = \sup\{x \in (0, \infty) : v(t, x) = (K - x)^+\}. \quad (4.101)$$

Proposition 4.8.2 *For each $t \in (0, \infty)$, $c(t) \in (0, K)$, $v(t, x) > (K - x)^+$ for all $x > c(t)$ and $v(t, x) = (K - x)^+$ for all $x \in [0, c(t)]$. The function $c : (0, \infty) \rightarrow (0, K)$ is non-increasing, continuous, and $\lim_{t \downarrow 0} c(t) = K$. We denote this last quantity by $c(0)$.*

PROOF. See Section 2.7 of [11]. $\square$

It turns out that v satisfies a parabolic partial differential equation in $\mathcal{C}$. In fact, $v(s, x)$ is once continuously differentiable in s and twice continuously differentiable in x for $(s, x) \in \mathcal{C}$. Then, for $(s, x) \in \mathcal{C}$, we let $v_s(s, x)$ denote the first partial derivative with respect to s and $v_x(s, x)$, $v_{xx}(s, x)$ respectively denote the first, second partial derivative with respect to x . We now motivate the form of the partial differential equation satisfied by v .

For each $\epsilon > 0$, let

$$\tau_\epsilon^* = \inf\{s \geq 0 : v(T - s, S_s) \leq (K - S_s)^+ + \epsilon\}.$$

Then, for each $\epsilon > 0$, by Itô's formula, we have P^* -a.s., for each $t \in [0, T]$,

$$\begin{aligned}
U_{t \wedge \tau_\epsilon^*} &= v(T - (t \wedge \tau_\epsilon^*), S_{t \wedge \tau_\epsilon^*}) \\
&= v(T, S_0) - \int_0^{t \wedge \tau_\epsilon^*} v_s(T - u, S_u) du + \int_0^{t \wedge \tau_\epsilon^*} v_x(T - u, S_u) dS_u \\
&\quad + \frac{1}{2} \int_0^{t \wedge \tau_\epsilon^*} v_{xx}(T - u, S_u) \sigma^2 S_u^2 du \\
&= v(T, S_0) + \int_0^{t \wedge \tau_\epsilon^*} v_x(T - u, S_u) \sigma S_u d\tilde{W}_u + \int_0^{t \wedge \tau_\epsilon^*} (Lv)(T - u, S_u) du,
\end{aligned}$$

where for $(s, x) \in \mathcal{C}$,

$$(Lv)(s, x) = -v_s(s, x) + rxv_x(s, x) + \frac{1}{2}\sigma^2 x^2 v_{xx}(s, x).$$

Hence, applying Itô's formula again, we have P^* -a.s., for each $t \in [0, T]$,

$$\begin{aligned}
U_{t \wedge \tau_\epsilon^*}^* &= e^{-r(t \wedge \tau_\epsilon^*)} U_{t \wedge \tau_\epsilon^*} \\
&= U_0^* + \int_0^{t \wedge \tau_\epsilon^*} e^{-ru} v_x(T - u, S_u) \sigma S_u d\tilde{W}_u \\
&\quad + \int_0^{t \wedge \tau_\epsilon^*} e^{-ru} (\tilde{L}v)(T - u, S_u) du,
\end{aligned}$$

where for $(s, x) \in \mathcal{C}$,

$$(\tilde{L}v)(s, x) = (Lv)(s, x) - rv(s, x). \quad (4.102)$$

Now, $\{U_{t \wedge \tau_\epsilon^*}^*, \mathcal{F}_t, t \in [0, T]\}$ is a right continuous martingale under P^* , and so it follows by the uniqueness of its Doob-Meyer decomposition that P^* -a.s., for all $t \in [0, T]$:

$$\int_0^{t \wedge \tau_\epsilon^*} e^{-ru} (\tilde{L}v)(T - u, S_u) du = 0.$$

One way in which this could be satisfied is if $\tilde{L}v = 0$ in $\mathcal{C}$, since $(T - u, S_u) \in \mathcal{C}$ for $u < \tau_\epsilon^* \leq \tau^*$. In fact, the following free boundary problem characterizes v , together with the boundary function c .

Proposition 4.8.3 *The pair $(f, d) = (v, c)$ is the unique pair of functions satisfying the following: $f : [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$ is continuous and such that $f(t, x)$ is once continuously differentiable in t and twice continuously differentiable in x for $(t, x) \in \mathcal{G} = \{(t, x) \in (0, \infty) \times (0, \infty) : x > d(t)\}$, $d : [0, \infty) \rightarrow$*

$(0, K]$ is non-increasing and left continuous, $d(0) = \lim_{t \downarrow 0} d(t) = K$,

$$\begin{aligned}
 \tilde{L}f &= 0 \quad \text{in } \mathcal{G}, \\
 f(t, x) &\geq (K - x)^+, \quad (t, x) \in [0, \infty) \times [0, \infty), \\
 f(t, x) &= K - x, \quad 0 \leq t < \infty, \quad 0 \leq x \leq d(t), \\
 f(0, x) &= (K - x)^+ \quad \text{for } d(0) \leq x < \infty, \\
 \lim_{x \rightarrow \infty} \max_{0 \leq s \leq t} |f(s, x)| &= 0, \quad t \in (0, \infty) \\
 \lim_{x \downarrow d(t)} f_x(t, x) &= -1, \quad t \in (0, \infty).
 \end{aligned}$$

PROOF. See Theorem 7.12 of Section 2.7 in [11]. $\square$

While the above proposition provides a characterization of v and the free boundary c , this still does not provide a closed form expression for v . However, the above free boundary value problem can be expressed in variational form and that can be used as the basis for a numerical method for approximating v . See [11], Section 2.8, for more discussion of this and other methods for approximating v .