

4.7 American Call Option

Consider an American call option with payoff process Y given by

$$Y_t = (S_t - K)^+, \quad t \in [0, T].$$

Note that by the Cauchy-Schwarz inequality and Doob's L^2 -martingale inequality,

$$\begin{aligned} E^{P^*} \left[\sup_{0 \leq t \leq T} Y_t \right] &\leq e^{rT} E^{P^*} \left[\sup_{0 \leq t \leq T} S_t^* \right] \\ &\leq e^{rT} \left(E^{P^*} \left[\sup_{0 \leq t \leq T} (S_t^*)^2 \right] \right)^{\frac{1}{2}} \\ &\leq 2e^{rT} \left(E^{P^*} [(S_T^*)^2] \right)^{\frac{1}{2}}, \\ &= 2e^{rT + \frac{1}{2}\sigma^2 T} \left(E^{P^*} \left[\exp \left(2\sigma \tilde{W}_T - \frac{1}{2}(4\sigma^2 T) \right) \right] \right)^{\frac{1}{2}} \\ &= 2e^{rT + \frac{1}{2}\sigma^2 T}. \end{aligned}$$

From the previous section, we know that the initial arbitrage free price is given by

$$U_0 = \text{ess sup}_{\tau \in \mathcal{T}_{[0, T]}} E^{P^*} [Y_\tau], \quad (4.88)$$

$$= \text{ess sup}_{\tau \in \mathcal{T}_{[0, T]}} E^{P^*} [e^{-r\tau} (S_\tau - K)^+]. \quad (4.89)$$

By part (ii) of Theorem 4.6.1, the essential supremum in the above expression is achieved by $\tau^* = \tau_0 = \inf\{t \in [0, T] : U_t^* = Y_t^*\}$. In general, neither U_0 nor τ^* is easy to compute. However, in the special case of the American call option, we will exploit the convexity and increasing property of the function $g(x) = (x - K)^+$, $x \geq 0$, to show that the initial arbitrage free price for an American call option is the same as that for the European call option with the same strike price and expiration time. One can also show that a buyer of the American call option who does not cash it in until the expiration time T will not create an arbitrage opportunity for the seller of the option. We leave this as an exercise for the reader.

We begin by deriving a stochastic integral equation for the discounted payoff process Y^* . Now, P^* -a.s., S satisfies the equation

$$S_t = S_0 + \int_0^t r S_s ds + \sigma \int_0^t S_s d\tilde{W}_s, \quad t \in [0, T]. \quad (4.90)$$

By the Itô-Tanaka formula (cf. Theorem VI.1.5 of Revuz-Yor [15]), P^* -a.s.,

$$Y_t = g(S_t) = g(S_0) + \int_0^t 1_{\{S_s > K\}} dS_s + \frac{1}{2} L_t, \quad t \in [0, T], \quad (4.91)$$

where $L = \{L_t, t \in [0, T]\}$ is the local time at K of the stock price process S . This local time is a continuous, non-decreasing, adapted process that measures the amount of time (in units of the quadratic variation of S) that S spends near K . Indeed, P^* -a.s.,

$$L_t = \lim_{\epsilon \rightarrow 0} \frac{1}{2\epsilon} \int_0^t 1_{(K-\epsilon, K+\epsilon)}(S_s) d[S]_s, \quad \text{for each } t \in [0, T], \quad (4.92)$$

where $d[S]_s = \sigma^2 S_s^2 ds$. (The Itô-Tanaka formula is a generalization of Itô's formula from twice continuously differentiable functions to convex functions of continuous semimartingales. It can be derived by passing to a limit in Itô's formula applied to a sequence of smooth functions $\{g_n\}$ that approximate g in a suitable manner. The term in Itô's formula containing the first derivative of such a function converges to the second term on the right side of (4.91). The term in Itô's formula containing the second derivative of such a function converges to the local time term in (4.91). See Chapter VI of [15] for details.)

Then, since $Y_t^* = e^{-rt} g(S_t)$, $t \in [0, T]$, we have P^* -a.s.,

$$\begin{aligned} Y_t^* &= Y_0^* - \int_0^t r e^{-rs} g(S_s) ds + \int_0^t e^{-rs} \left(1_{\{S_s > K\}} dS_s + \frac{1}{2} dL_s \right) \\ &= Y_0^* - \int_0^t r e^{-rs} g(S_s) ds + \int_0^t e^{-rs} \left(1_{\{S_s > K\}} (r S_s ds + \sigma S_s d\tilde{W}_s) + \frac{1}{2} dL_s \right) \\ &= Y_0^* + \int_0^t e^{-rs} 1_{\{S_s > K\}} \sigma S_s d\tilde{W}_s \\ &\quad + \int_0^t e^{-rs} (1_{\{S_s > K\}} r S_s - r g(S_s)) ds + \frac{1}{2} \int_0^t e^{-rs} dL_s. \end{aligned}$$

Notice that by the definition of g , we have $1_{\{S_s > K\}} r S_s - r g(S_s) = 1_{\{S_s > K\}} r K$ for all $s \in [0, T]$. Thus, P^* -a.s., for $t \in [0, T]$,

$$Y_t^* = Y_0^* + \sigma \int_0^t e^{-rs} 1_{\{S_s > K\}} S_s d\tilde{W}_s + r K \int_0^t e^{-rs} 1_{\{S_s > K\}} ds + \frac{1}{2} \int_0^t e^{-rs} dL_s.$$

The first integral above defines a continuous local martingale and the final two terms define continuous, adapted, non-decreasing processes. Thus $Y^* = \{Y_t^*, \mathcal{F}_t, t \in [0, T]\}$ is a continuous local submartingale. However, since Y_t^* is dominated by the P^* -integrable random variable $\sup_{t \in [0, T]} Y_t$ for each $t \in [0, T]$, Y^* is in fact a submartingale. Therefore, by Doob's stopping theorem for continuous submartingales, for each $\tau \in \mathcal{T}_{[0, T]}$,

$$E^{P^*}[Y_\tau^*] \leq E^{P^*}[Y_T^*].$$

This together with (4.88) gives

$$U_0 = \text{ess sup}_{\tau \in \mathcal{T}_{[0, T]}} E^{P^*}[Y_\tau^*] \leq E^{P^*}[Y_T^*].$$

But notice that $\tau \equiv T$ is an element of $\mathcal{T}_{[0,T]}$ and so by (4.88),

$$U_0 \geq E^{P^*} [Y_T^*].$$

It follows that the initial arbitrage free price for the American call option with strike price K and expiration time T is given by

$$U_0 = E^{P^*} [Y_T^*] = e^{-rT} E^{P^*} \left[(S_T - K)^+ \right],$$

which is the same as the initial arbitrage free price of the European call option with the same strike price and expiration time.