

Note that the Black-Scholes formula (4.67) does not involve the rate of return μ of the stock. The only stock price parameter that appears in the formula is the volatility σ . Given this volatility, and assuming the Black-Scholes model for the stock price, one can use the Black-Scholes formula to price a European call option with a given strike price K and expiration time T . For this application, one might hope to use historical data to estimate the volatility parameter. However, such estimates typically produce biased option prices. Another method is often used to estimate volatility, namely, the Black-Scholes formula is used in reverse to obtain an estimate of the volatility of the underlying stock price process from market prices of frequently traded options, e.g., those traded on an exchange. An estimate of volatility obtained in this way is called *implied volatility*. This can then be used as an input to the Black-Scholes model to determine prices of other European contingent claims that are not traded frequently.

All European call options based on a given stock use the same volatility parameter σ for their pricing, regardless of the strike price or the expiration date of the option. Empirical plots of implied volatility as a function of strike price often have a U-shape or half U-shape, referred to as a *smile* or *smirk*. Various efforts have been made to generalize the Black-Scholes model to provide a better fit of implied volatility to theoretical volatility. See Musiela and Rutkowski [13], Chapter 6, for a discussion of this topic.

4.6 Pricing an American Contingent Claim

An *American contingent claim* (ACC) is represented by a non-negative, continuous, adapted stochastic process $Y = \{Y_t, t \in [0, T]\}$ evolving in continuous time such that $E^{P^*} [\sup_{0 \leq t \leq T} Y_t] < \infty$. The random variable Y_t , $t \in [0, T]$, is interpreted as the payoff for the claim if the owner cashes it in at time t . The time at which the owner cashes in the claim is required to be a stopping time taking values in $[0, T]$, i.e., it is a random variable $\tau : \Omega \rightarrow [0, T]$ such that $\{\tau \leq t\} \in \mathcal{F}_t$, for each $t \in [0, T]$. For $0 \leq s \leq t \leq T$, let $\mathcal{T}_{[s, t]}$ denote the set of stopping times that take values in the interval $[s, t]$. An example of an American contingent claim is an *American call option* with strike price K , which has payoff $Y_t = (S_t - K)^+$ at time t , $t \in [0, T]$. Note that if $S_t \leq K$, cashing in the contingent claim at time t is equivalent to not exercising the option at all. We have adopted this convention so that we can use one framework for treating all contingent claims, including options and contracts.

As noted in Section 2.3, devoted to pricing American contingent claims in the setting of the discrete binomial model, an important feature of an American contingent claim is that the buyer and the seller of such a derivative have different actions available to them — the buyer may cash in the claim at any stopping time $\tau \in \mathcal{T}_{[0, T]}$, whereas the seller seeks protection from the risk associated with all possible choices of the stopping time τ by the buyer. Also, as for the binomial model, in pricing and hedging American contingent claims, a critical role

is played by a superhedging (or perfect hedging) strategy that hedges the risk for the seller of an American contingent claim.

A *superhedging strategy* for the seller of an American contingent claim, $Y = \{Y_t, t \in [0, T]\}$, is an admissible strategy $\phi = \{(\alpha_t, \beta_t) : t \in [0, T]\}$ with value $V_t(\phi)$ at time $t \in [0, T]$, such that $V_t(\phi) \geq Y_t$ for each $t \in [0, T]$, P^* -a.s. In the context of the binomial model, such a ϕ can be constructed stepwise by proceeding backwards through the binary tree. In continuous time, such as stepwise construction is no longer available. Nevertheless, one can exploit the theory of Snell envelopes to show that a superhedging strategy exists.

For a collection of non-negative random variables $\{X_\gamma, \gamma \in \Gamma\}$ defined on a probability space, their *essential supremum* is a random variable X satisfying

- (i) $X \geq X_\gamma$ a.s. for each $\gamma \in \Gamma$, and
- (ii) if Z is a random variable such that $Z \geq X_\gamma$ a.s. for each $\gamma \in \Gamma$, then $Z \geq X$ a.s.

We write $X = \text{esssup}_{\gamma \in \Gamma} X_\gamma$. (For the existence and properties of the essential supremum, see Karatzas and Shreve [11], Appendix A.)

For each $t \in [0, T]$, let $Y_t^* = \frac{Y_t}{B_t} = e^{-rt} Y_t$. Define a process $U^* = \{U_t^*, t \in [0, T]\}$ as follows. For each $t \in [0, T]$, let

$$U_t^* = \text{esssup}_{\tau \in \mathcal{T}_{[t, T]}} E^{P^*} [Y_\tau^* \mid \mathcal{F}_t], \quad (4.79)$$

where we take U_t^* to be $\mathcal{F}_t$ -measurable, without loss of generality. The right member above is finite P^* -a.s. since we assumed that $Y_t \geq 0$ for each $t \in [0, T]$ and $E^{P^*}[\sup_{0 \leq s \leq T} Y_s] < \infty$. It is straightforward to verify that $\{U_t^*, \mathcal{F}_t, t \in [0, T]\}$ is a supermartingale. In fact (cf. [11], Appendix D), there is a version of this supermartingale whose paths are right continuous with finite left limits (abbreviated as r.c.l.l.). Henceforth, we shall assume that U^* is such a “nice” version. Let $\tau^*(t) \equiv \inf\{s \geq t : U_s^* = Y_s^*\} \wedge T$ for each $t \in [0, T]$ and define $\tau^* = \tau^*(0)$. Note that $U_{\tau^*}^* = Y_{\tau^*}^*$ P^* -a.s., since U^* has right continuous paths, Y^* has continuous paths, and $U_T^* = Y_T^*$ P^* -a.s.

The following theorem is proved in Appendix D of Karatzas and Shreve [11].

Theorem 4.6.1

- (i) $\{U_t^*, \mathcal{F}_t, t \in [0, T]\}$ is the smallest P^* -supermartingale that has r.c.l.l. paths and satisfies P^* -a.s. $U_t^* \geq Y_t^*$ for each $t \in [0, T]$,
- (ii) $\tau^*(t) \in \mathcal{T}_{[t, T]}$ and

$$U_t^* = E^{P^*} [Y_{\tau^*(t)}^* \mid \mathcal{F}_t], \quad t \in [0, T],$$

(iii) $\{U_{t \wedge \tau^*}^*, \mathcal{F}_t, t \in [0, T]\}$ is a martingale under P^* .

By the Doob-Meyer decomposition theorem (cf. [11], Theorem D.13), we have the following decomposition for the uniformly integrable supermartingale U^* :

$$U_t^* = U_0^* + M_t - A_t, \quad t \in [0, T], \quad (4.80)$$

where $\{M_t, \mathcal{F}_t, t \in [0, T]\}$ is a P^* -martingale with r.c.l.l. paths, $M_0 = 0$, $\{A_t, \mathcal{F}_t, t \in [0, T]\}$ is an increasing, adapted process such that $A_0 = 0$, $E^{P^*}[A_t] < \infty$ for each $t \in [0, T]$, and A is predictable (i.e., $\{A(t, \omega) : (t, \omega) \in [0, T] \times \Omega\}$ is measurable with respect to the σ -algebra on $[0, T] \times \Omega$ generated by the left continuous adapted processes). In fact, this decomposition is unique (up to indistinguishability of stochastic processes). By the martingale representation theorem (Theorem 4.3.1), there is a $(\mathcal{B}_T \times \mathcal{F}_T)$ -measurable, adapted process $\eta = \{\eta_t : t \in [0, T]\}$ such that P^* -a.s., $\int_0^T \eta_s^2 ds < \infty$ and

$$M_t = \int_0^t \eta_s d\tilde{W}_s, \quad \text{for all } t \in [0, T]. \quad (4.81)$$

In particular, M has continuous paths P^* -a.s.

Let

$$\phi_t = (\alpha_t, \beta_t) = \left(\frac{\eta_t}{\sigma S_t^*}, U_0^* + M_t - \frac{\eta_t}{\sigma} \right), \quad t \in [0, T]. \quad (4.82)$$

Then, ϕ is a trading strategy since it satisfies the measurability and local integrability properties required of a trading strategy (cf. Section 4.1). The discounted value process for ϕ satisfies P^* -a.s. for each $t \in [0, T]$:

$$\begin{aligned} V_t^*(\phi) &= \frac{\eta_t}{\sigma} + U_0^* + M_t - \frac{\eta_t}{\sigma} \\ &= U_0^* + M_t \\ &= U_0^* + \int_0^t \eta_s d\tilde{W}_s \\ &= U_0^* + \int_0^t \alpha_s \sigma S_s^* d\tilde{W}_s \\ &= V_0(\phi) + \int_0^t \alpha_s dS_s^*, \end{aligned} \quad (4.83)$$

where the third line is by (4.81), the fourth is by (4.82), and the fifth is by (4.30). We also used the fact that $U_0^* = V_0^*(\phi) = V_0(\phi)$. It follows that ϕ is self-financing. Moreover, since U_0^* is a finite constant and M is a martingale under P^* , $\{V_t^*(\phi), \mathcal{F}_t, t \in [0, T]\}$ is a P^* -martingale. Hence, ϕ is an admissible strategy.

Next we verify that ϕ is a superhedging strategy. For this note that by (4.83), (4.80) and (i) of Theorem 4.6.1, we have P^* -a.s.,

$$V_t^*(\phi) = U_0^* + M_t = U_t^* + A_t \geq U_t^* \geq Y_t^*, \quad t \in [0, T]. \quad (4.84)$$

Finally, by Theorem 4.6.1, (4.80), and the uniqueness of the Doob-Meyer decomposition, P^* -a.s.,

$$U_{t \wedge \tau^*}^* = U_0^* + M_{t \wedge \tau^*} = V_{t \wedge \tau^*}^*(\phi), \quad t \in [0, T].$$

We now argue that U_0 is the unique arbitrage free initial price for the American contingent claim at time zero. For this, we need the notion of an arbitrage in a market where the stock and bond can be traded, and the American contingent claim (ACC) can be bought or sold at time zero. For such a market, let the initial price of the ACC be a constant C_0 . There are two types of arbitrage opportunities, one for a seller and another for a buyer of the ACC. The *seller* of the ACC has an *arbitrage opportunity* if there is an admissible strategy ϕ^s in stock and bond such that $V_0(\phi^s) = C_0$ and for all stopping times $\tau \in \mathcal{T}_{[0, T]}$,

$$V_\tau(\phi^s) - Y_\tau \geq 0 \quad \text{and} \quad E^{P^*} [V_\tau(\phi^s) - Y_\tau] > 0. \quad (4.85)$$

The *buyer* of the ACC has an *arbitrage opportunity* if there is an admissible strategy ϕ^b such that $V_0(\phi^b) = -C_0$ and there exists a stopping time $\tau \in \mathcal{T}_{[0, T]}$ such that

$$V_\tau(\phi^b) + Y_\tau \geq 0 \quad \text{and} \quad E^{P^*} [V_\tau(\phi^b) + Y_\tau] > 0. \quad (4.86)$$

In conditions (4.85)–(4.86), one could equivalently evaluate the expectations under P , instead of P^* , since P and P^* have the same sets of probability zero. The initial price C_0 is arbitrage free if there is no arbitrage opportunity for a seller or buyer of the contingent claim at this initial price.

To take advantage of a seller's arbitrage opportunity, an investor could sell one ACC at time zero for C_0 and invest the proceeds C_0 according to the trading strategy ϕ^s until the claim is cashed in by the buyer at some stopping time τ . At time τ , the seller would give the buyer Y_τ to payoff the claim and could put the remainder of his wealth $V_\tau(\phi^s) - Y_\tau$ in bond for the time period $(\tau, T]$. This would result in a final value of $(V_\tau(\phi^s) - Y_\tau)e^{r(T-\tau)}$ which is non-negative and is strictly positive with positive probability under P^* .

To take advantage of a buyer's arbitrage opportunity, an investor could buy one ACC at time zero for C_0 , and invest $-C_0$ according to ϕ^b until the time τ when the buyer cashes in the claim. The buyer would then have $V_\tau(\phi^b) + Y_\tau$ at time τ and could put this in bond for the time period $(\tau, T]$ so that the buyer's final wealth would be $(V_\tau(\phi^b) + Y_\tau)e^{r(T-\tau)}$ which is non-negative and is strictly positive with positive probability under P^* .

Theorem 4.6.2 *The unique arbitrage free price at time zero for the American contingent claim is U_0 .*

PROOF. We first show that the arbitrage free price cannot be anything other than U_0 , i.e., we establish uniqueness of an arbitrage free initial price. Let ϕ^* be the superhedging strategy defined in (4.82).

Suppose that $C_0 > U_0$. Then there is an arbitrage opportunity for the seller of the American contingent claim. To see this, let ϕ^s denote the self-financing trading strategy in stock and bond corresponding to investing U_0 according to the superhedging strategy ϕ^* and $C_0 - U_0$ in bonds for all time. Note that the initial value $V_0(\phi^s) = C_0$, and, by (4.84), the value $V_t(\phi^s)$ of ϕ^s at time t is at least as great as Y_t for each $t \in [0, T]$. Then for any stopping time $\tau \in \mathcal{T}_{[0, T]}$,

$$\begin{aligned} V_\tau(\phi^s) - Y_\tau &= V_\tau(\phi^*) + (C_0 - U_0)B_\tau - Y_\tau \\ &\geq Y_\tau + (C_0 - U_0)B_\tau - Y_\tau \\ &\geq (C_0 - U_0)B_\tau, \end{aligned}$$

where $(C_0 - U_0)B_\tau > 0$. Thus, there is an arbitrage opportunity for a seller of the American contingent claim.

On the other hand, suppose that $C_0 < U_0$. Then there is an arbitrage opportunity for the buyer of the American contingent claim. To see this, let ϕ^b denote the self-financing trading strategy corresponding to investing $-U_0$ according to the negative $-\phi^*$ of the superhedging strategy ϕ^* and investing $U_0 - C_0$ in bonds for all time. Furthermore, consider the stopping time τ^* (viewed as the time at which an ACC, bought initially by the buyer, should be cashed in). Note that $V_t(-\phi^*) = -V_t(\phi^*)$ for $t \in [0, T]$, and $V_0(\phi^b) = -C_0$. Then, using the fact that $V_{\tau^*}(\phi^*) = U_{\tau^*} = Y_{\tau^*}$, we have

$$V_{\tau^*}(\phi^b) + Y_{\tau^*} = -V_{\tau^*}(\phi^*) + (U_0 - C_0)B_{\tau^*} + Y_{\tau^*} = (U_0 - C_0)B_{\tau^*}.$$

Since $(U_0 - C_0)B_{\tau^*} > 0$, there is an arbitrage opportunity for a buyer of the American contingent claim.

Finally, suppose that $C_0 = U_0$. We need to show that $C_0 = U_0$ is arbitrage free, i.e., that there exists an arbitrage free initial price. We begin by showing that there is no arbitrage opportunity for a seller of an American contingent claim with $C_0 = U_0$. For a proof by contradiction, suppose that there exists an admissible strategy ϕ^s such that $V_0(\phi^s) = U_0$ and for each $\tau \in \mathcal{T}_{[0, T]}$, (4.85) holds. Note that $\tau^* \in \mathcal{T}_{[0, T]}$. Then it follows from (4.85), with $\tau = \tau^*$, that $V_{\tau^*}(\phi^s) - Y_{\tau^*} \geq 0$ and strict inequality holds with positive probability under P^* . Multiplying through by $e^{-r\tau^*}$, and taking expectations, we see that

$$E^{P^*} [V_{\tau^*}^*(\phi^s) - Y_{\tau^*}^*] > 0. \quad (4.87)$$

On the other hand, by Doob's stopping theorem for continuous martingales (cf. [4]), $E^{P^*} [V_{\tau^*}^*(\phi^s)] = V_0^*(\phi^s) = U_0^*$, and by (ii) in Theorem 4.6.1, $E^{P^*} [Y_{\tau^*}^*] = U_0^*$. Combining these two properties, we obtain $E^{P^*} [V_{\tau^*}^*(\phi^s) - Y_{\tau^*}^*] = 0$, which contradicts (4.87). Therefore, no such ϕ^s exists and consequently there is no arbitrage opportunity for a seller of the American contingent claim.

Next we show that there is no arbitrage opportunity for a buyer of the American contingent claim with $C_0 = U_0$. For a proof by contradiction, suppose that there exists an admissible strategy ϕ^b such that $V_0(\phi^b) = -U_0$ and a stopping time

$\tau \in \mathcal{T}_{[0,T]}$ such that (4.86) holds. Then, $E^{P^*} [V_\tau^*(\phi^b) + Y_\tau^*] > 0$. However, by Doob's stopping theorem, $E^{P^*} [V_\tau^*(\phi^b)] = V_0^*(\phi^b) = -U_0^*$, and, by (ii) in Theorem 4.6.1, $E^{P^*} [Y_\tau^*] \leq U_0^*$. Therefore, $E^{P^*} [V_\tau^*(\phi^b) + Y_\tau^*] \leq 0$, which yields the desired contradiction. $\square$