

4.4 Arbitrage Free Price Process

Consider a European contingent claim $X \in \mathcal{F}_T$ such that $E^{P^*}[|X|] < \infty$. To determine the arbitrage free price process for this claim, we need a notion of admissible strategy for trading in stock, bond and the contingent claim.

We assume that any given price process $C = \{C_t, t \in [0, T]\}$ for the European contingent claim is a right continuous, adapted process, where C_t represents the price of the European contingent claim at time t . (Adaptedness is the minimal assumption we would expect on this process and right continuity is a reasonable regularity assumption.) We suppose that for a given price process C , there is a set of admissible strategies Ψ_C for trading in stock, bond and the contingent claim. At a minimum, an element ψ of Ψ_C should be a three-dimensional adapted process $\psi = \{\psi_t = (\alpha_t, \beta_t, \gamma_t), t \in [0, T]\}$.

To show uniqueness of the price process, we do not want to place unnecessary restrictions on C just so that it can be used as an integrator (in defining self-financing trading strategies). In fact, to show uniqueness, we shall only need the reasonable assumption that rather simple strategies of the following type are admissible: those where one buys or sells one contingent claim, or does nothing, at some deterministic time t^* , and then holds that position in the claim until time T , but where one may trade as usual in the stock and bond over $[t^*, T]$. More precisely, we assume that strategies of the following form are in Ψ_C :

$$\psi_t(\omega) = 1_{\{t \geq t^*\}} 1_A(\omega) (\tilde{\alpha}_t(\omega), \tilde{\beta}_t(\omega) + \tilde{\delta}_t(\omega), \tilde{\gamma}_t(\omega)), \quad \omega \in \Omega, \quad t \in [0, T], \quad (4.48)$$

are admissible, where t^* is a fixed time in $[0, T]$, $A \in \mathcal{F}_{t^*}$, $\{(\tilde{\alpha}_t, \tilde{\beta}_t), t \in [0, T]\}$ is an admissible self-financing trading strategy in the primary (stock-bond) market, $\tilde{\gamma}_t = +1$ for all t or $\tilde{\gamma}_t = -1$ for all t , $\tilde{\delta}_t = \tilde{\delta}_{t^*}$ for all t is chosen such that $\tilde{\alpha}_{t^*} S_{t^*} + (\tilde{\beta}_{t^*} + \tilde{\delta}_{t^*}) B_{t^*} + \tilde{\gamma}_{t^*} C_{t^*} = 0$. Under such a strategy, the investor does nothing prior to the time t^* . On A^c , the investor continues to do nothing for the remaining interval $[t^*, T]$. On A , if $\tilde{\gamma}_{t^*} \equiv +1$, at t^* the investor buys one contingent claim for C_{t^*} and invests $-C_{t^*}$ in the stock-bond market according to $(\tilde{\alpha}_t, \tilde{\beta}_t + \tilde{\delta}_t)$ for $t \in [t^*, T]$. On A , if $\tilde{\gamma}_{t^*} \equiv -1$, instead of buying one contingent claim at time t^* , the investor sells one contingent claim at t^* and invests the proceeds C_{t^*} according to $(\tilde{\alpha}_t, \tilde{\beta}_t + \tilde{\delta}_t)$ for $t \in [t^*, T]$. Since the value of the strategy ψ is zero at time t^* and $\tilde{\delta}_t$ and $\tilde{\gamma}_t$ do not vary with $t \geq t^*$, the strategy will be naturally self-financing, since $(\tilde{\alpha}, \tilde{\beta})$ is assumed to have this property.

We will prove below that when the price process C is a continuous version of

$$e^{rt} E^{P^*}[X^* | \mathcal{F}_t], \quad t \in [0, T], \quad (4.49)$$

it is an arbitrage free price process. For this, we need similar constraints on the admissible strategies to what we had in the case of the primary market, to ensure that doubling type strategies are ruled out. Note that in this case, if $\phi^* = (\alpha^*, \beta^*)$ is a replicating strategy for X , then $C_t^* = e^{-rt} C_t = V_t^*(\phi^*)$

defines a martingale under P^* and P^* -a.s.,

$$dC_t^* = dV_t^*(\phi^*) = \alpha_t^* dS_t^* = \sigma \alpha_t^* S_t^* d\tilde{W}_t. \quad (4.50)$$

Consequently, in this case, we let the class of admissible strategies Ψ_C be the set of three-dimensional processes $\psi = (\alpha, \beta, \gamma)$ such that

- (i) $\psi : [0, T] \times \Omega \rightarrow \mathbb{R}^3$ is $(\mathcal{B}_T \times \mathcal{F}_T)$ -measurable, where $\psi(t, \omega) = \psi_t(\omega)$,
- (ii) ψ is adapted, i.e., $\psi_t \in \mathcal{F}_t$ for each $t \in [0, T]$,
- (iii) $\int_0^T \alpha_t^2 dt < \infty$, $\int_0^T |\beta_t| dt < \infty$, $\int_0^T \gamma_t^2 (\alpha_t^*)^2 dt < \infty$, P^* -a.s.,
- (iv) P^* -a.s., for each $t \in [0, T]$,

$$V_t(\psi) \equiv \alpha_t S_t + \beta_t B_t + \gamma_t C_t \quad (4.51)$$

$$= V_0(\psi) + \int_0^t \alpha_s dS_s + \int_0^t \beta_s dB_s + \int_0^t \gamma_s dC_s, \quad (4.52)$$

- (v) $\{V_t^*(\psi) = e^{-rt} V_t(\psi), \mathcal{F}_t, t \in [0, T]\}$ is a martingale under P^* .

Properties (i)–(iii) above ensure that the integrals appearing in (4.52) are well defined, property (iv) is the self-financing property of ψ , and (v) amounts to a constraint on the value of ψ to rule out doubling strategies. Note that since S^* and C^* are already martingales under P^* , this last condition amounts to some control on the size of ψ . One can verify that strategies of the form (4.48) satisfy the above properties (i)–(v) when C is a continuous version of (4.49).

Given a price process C for the contingent claim, an *arbitrage opportunity* in the stock-bond-contingent claim market is an admissible strategy $\psi \in \Psi_C$ such that $V_0(\psi) = 0$, $V_T(\psi) > 0$ and $E^{P^*}[V_T(\psi)] > 0$.

In the following theorem, uniqueness means uniqueness up to indistinguishability of stochastic processes.

Theorem 4.4.1 *The unique arbitrage free price process for the P^* -integrable European contingent claim X is a continuous version of $\{e^{rt} E^{P^*}[X^* | \mathcal{F}_t], t \in [0, T]\}$.*

PROOF. Consider a price process $\{C_t, t \in [0, T]\}$ for the European contingent claim. By Theorem 4.3.2, there is a replicating strategy $\phi^* = (\alpha^*, \beta^*)$ for X and its value process $\{V_t(\phi^*), t \in [0, T]\}$ is a continuous version of $\{e^{rt} E^{P^*}[X^* | \mathcal{F}_t], t \in [0, T]\}$. Suppose that $P^*(C_t \neq V_t(\phi^*) \text{ for some } t \in [0, T]) > 0$. Since the process C is assumed to be right continuous and $V(\phi^*)$ is continuous, there exists a $t^* \in [0, T]$ such that $P^*(C_{t^*} \neq V_{t^*}(\phi^*)) > 0$. Let $A = \{\omega : C_{t^*}(\omega) > V_{t^*}(\phi^*)(\omega)\}$ and $\tilde{A} = \{\omega : C_{t^*}(\omega) < V_{t^*}(\phi^*)(\omega)\}$. Then either $P^*(A) > 0$ or $P^*(\tilde{A}) > 0$. First suppose that $P^*(A) > 0$ and suppose that

the replicating strategy has the form $\phi^* = (\alpha^*, \beta^*)$. Define $\psi = \{(\alpha_t, \beta_t, \gamma_t), t \in [0, T]\}$ by

$$\psi_t(\omega) = \begin{cases} (0, 0, 0), & t < t^*, \\ (0, 0, 0), & t \geq t^*, \omega \in A^c, \\ \left(\alpha_t^*(\omega), \beta_t^*(\omega) + \frac{C_{t^*}(\omega) - V_{t^*}(\phi^*)(\omega)}{B_{t^*}}, -1 \right), & t \geq t^*, \omega \in A, \end{cases} \quad (4.53)$$

or in other words, for $t \in [0, T]$, $\omega \in \Omega$,

$$\psi_t(\omega) = 1_{\{t \geq t^*\}} 1_A(\omega) \left(\alpha_t^*, \beta_t^* + \frac{C_{t^*} - V_{t^*}(\phi^*)}{B_{t^*}}, -1 \right) (\omega). \quad (4.54)$$

Clearly, the value of the portfolio represented by ψ_t is zero for $t \leq t^*$, and so the value at time zero is zero. Moreover, the value at time T is zero on A^c . Also, since ϕ^* is a replicating strategy for X , on A , the value at time T is $\left(\frac{C_{t^*} - V_{t^*}(\phi^*)}{B_{t^*}} \right) B_T$, which is strictly positive. Since $P^*(A) > 0$, ψ is an arbitrage opportunity.

Similarly, if $P^*(\tilde{A}) > 0$, then ψ defined for $t \in [0, T]$, $\omega \in \Omega$ by

$$\psi_t(\omega) = 1_{\{t \geq t^*\}} 1_{\tilde{A}}(\omega) \left(-\alpha_t^*, -\beta_t^* + \frac{V_{t^*}(\phi^*) - C_{t^*}}{B_{t^*}}, 1 \right) (\omega), \quad (4.55)$$

is an arbitrage opportunity.

Thus, we have shown that the only possible arbitrage free price process (up to indistinguishability) is given by $C_t = V_t(\phi^*)$, $t \in [0, T]$. Next we show that this price process is arbitrage free. For a contradiction, suppose that with this price process, there is $\psi \in \Psi_C$ such that $V_0(\psi) = 0$, $V_T(\psi) \geq 0$ and $E^{P^*}[V_T(\psi)] > 0$. Now, by the admissibility assumption on ψ , the discounted value process $V^*(\psi)$ is a martingale under P^* and so

$$0 = V_0(\psi) = E^{P^*}[V_T^*(\psi)] = e^{-rT} E^{P^*}[V_T(\psi)] > 0, \quad (4.56)$$

a contradiction. Thus, there cannot be an arbitrage opportunity with this price process. $\square$

4.5 European Call Options

In this section, we apply the results of the previous section to compute the arbitrage free price process for a European call option and we also identify a replicating strategy for the option.

Let $X = (S_T - K)^+$ be a European call option with strike price K and expiration time T . Notice that $0 \leq X \leq S_T$ and $E^{P^*}[S_T] = S_0 e^{rt} < \infty$. Hence, by

Theorem 4.4.1, the arbitrage free price process $\{C_t, t \in [0, T]\}$ is a continuous adapted process such that for each $t \in [0, T]$, if $C_t^* = C_t e^{-rt}$, then we have P^* -a.s.,

$$C_t^* = E^{P^*} [X^* | \mathcal{F}_t] = E^{P^*} \left[(S_T^* - K^*)^+ | \mathcal{F}_t \right], \quad (4.57)$$

where $K^* = e^{-rT} K$. It follows from (4.29) that P^* -a.s., for each $t \in [0, T]$,

$$S_T^* = S_t^* \frac{S_0 \exp \left(\sigma \tilde{W}_T - \frac{1}{2} \sigma^2 T \right)}{S_0 \exp \left(\sigma \tilde{W}_t - \frac{1}{2} \sigma^2 t \right)} = S_t^* \exp \left(\sigma \left(\tilde{W}_T - \tilde{W}_t \right) - \frac{1}{2} \sigma^2 (T - t) \right). \quad (4.58)$$

On substituting this into the right member of (4.57), we obtain that for $t \in [0, T]$, P^* -a.s.,

$$C_t^* = E^{P^*} \left[\left(S_t^* \exp \left(\sigma \left(\tilde{W}_T - \tilde{W}_t \right) - \frac{1}{2} \sigma^2 (T - t) \right) - K^* \right)^+ | \mathcal{F}_t \right]. \quad (4.59)$$

Using the fact that $S_t^* \in \mathcal{F}_t$ and that $\tilde{W}_T - \tilde{W}_t$ is a normal random variable with mean zero and variance $T - t$ that is independent of $\mathcal{F}_t$ under P^* (since $\mathcal{F}_t$ is generated by $\tilde{W}_{\cdot \wedge t}$ and the P^* -null sets), it follows that for $t \in [0, T]$, P^* -a.s.,

$$C_t^* = \frac{1}{\sqrt{2\pi(T-t)}} \int_{-\infty}^{\infty} \left(S_t^* \exp \left(\sigma y - \frac{1}{2} \sigma^2 (T - t) \right) - K^* \right)^+ \exp \left(\frac{-y^2}{2(T-t)} \right) dy,$$

and $C_T^* = (S_T^* - K^*)^+$. Let $z = y/\sqrt{T-t}$. Then the expression on the right above becomes

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(S_t^* \exp \left(\sigma z \sqrt{T-t} - \frac{1}{2} \sigma^2 (T - t) \right) - K^* \right)^+ \exp \left(-\frac{1}{2} z^2 \right) dz. \quad (4.61)$$

Define $f : [0, T] \times (0, \infty) \rightarrow [0, \infty)$ by $f(T, x) = (x - K^*)^+$ for all $x \in (0, \infty)$, and for $t \in [0, T)$, $x \in (0, \infty)$,

$$f(t, x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(x \exp \left(\sigma z \sqrt{T-t} - \frac{1}{2} \sigma^2 (T - t) \right) - K^* \right)^+ \exp \left(-\frac{1}{2} z^2 \right) dz. \quad (4.62)$$

Then, by the above, for each $t \in [0, T]$, P^* -a.s.,

$$C_t^* = f(t, S_t^*). \quad (4.63)$$

For $t \in [0, T)$ and $x \in (0, \infty)$, let $l_t(x)$ be such that the integrand in (4.62) is positive if and only if $z > l_t(x)$, i.e.,

$$l_t(x) = \frac{\log \left(\frac{K^*}{x} \right)}{\sigma \sqrt{T-t}} + \frac{1}{2} \sigma \sqrt{T-t}. \quad (4.64)$$

Thus, for $t \in [0, T)$ and $x \in (0, \infty)$,

$$\begin{aligned}
f(t, x) &= \frac{x}{\sqrt{2\pi}} \int_{l_t(x)}^{\infty} \exp\left(\sigma z \sqrt{T-t} - \frac{1}{2}\sigma^2(T-t) - \frac{z^2}{2}\right) dz \\
&\quad - \frac{K^*}{\sqrt{2\pi}} \int_{l_t(x)}^{\infty} \exp\left(\frac{-z^2}{2}\right) dz \\
&= \frac{x}{\sqrt{2\pi}} \int_{l_t(x)}^{\infty} \exp\left(\frac{-(z - \sigma\sqrt{T-t})^2}{2}\right) dz - K^* \Phi(-l_t(x)) \\
&= \frac{x}{\sqrt{2\pi}} \int_{l_t(x) - \sigma\sqrt{T-t}}^{\infty} \exp\left(\frac{-u^2}{2}\right) du - K^* \Phi(-l_t(x)) \\
&= x\Phi\left(-l_t(x) + \sigma\sqrt{T-t}\right) - K^* \Phi(-l_t(x)).
\end{aligned}$$

Here we have used the change of variable $u = z - \sigma\sqrt{T-t}$ and the symmetry of Φ , where Φ is the cumulative distribution function for a standard normal random variable with mean zero and variance one:

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z \exp\left(\frac{-u^2}{2}\right) du, \quad z \in (-\infty, \infty). \quad (4.65)$$

Thus, for $t \in [0, T)$, P^* -a.s.,

$$C_t^* = f(t, S_t^*) = S_t^* \Phi\left(\frac{\log\left(\frac{S_t^*}{K^*}\right)}{\sigma\sqrt{T-t}} + \frac{1}{2}\sigma\sqrt{T-t}\right) - K^* \Phi\left(\frac{\log\left(\frac{S_t^*}{K^*}\right)}{\sigma\sqrt{T-t}} - \frac{1}{2}\sigma\sqrt{T-t}\right), \quad (4.66)$$

and in particular, P^* -a.s.,

$$C_0 = C_0^* = S_0 \Phi\left(\frac{\log\left(\frac{S_0}{K^*}\right)}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}\right) - K^* \Phi\left(\frac{\log\left(\frac{S_0}{K^*}\right)}{\sigma\sqrt{T}} - \frac{1}{2}\sigma\sqrt{T}\right), \quad (4.67)$$

which is the famous *Black-Scholes formula*.

The Black-Scholes formula gives the arbitrage free price for the European call option, but it does not immediately give a replicating or hedging strategy for the option. Recall that the existence of such a hedging strategy was guaranteed by the martingale representation theorem (cf. the proof of Theorem 4.4.1), but that result is non-constructive. We will now use stochastic calculus and the function f defined above to obtain such a hedging strategy.

Note that $f : [0, T] \times (0, \infty)$ is defined for $t \in [0, T)$, $x \in (0, \infty)$ by

$$f(t, x) = x\Phi\left(\frac{\log\left(\frac{x}{K^*}\right)}{\sigma\sqrt{T-t}} + \frac{1}{2}\sigma\sqrt{T-t}\right) - K^* \Phi\left(\frac{\log\left(\frac{x}{K^*}\right)}{\sigma\sqrt{T-t}} - \frac{1}{2}\sigma\sqrt{T-t}\right),$$

and

$$f(T, x) = (x - K^*)^+. \quad (4.68)$$

It is straightforward to verify from the above formula that $f \in C^{1,2}([0, T] \times (0, \infty))$, i.e., as a function of $(t, x) \in [0, T] \times (0, \infty)$, $f(t, x)$ is once continuously differentiable in t and twice continuously differentiable in x . In particular, the first partial derivative with respect to x is given for $t \in [0, T)$, $x \in (0, \infty)$, by

$$f_x(t, x) = \Phi \left(\frac{\log \left(\frac{x}{K^*} \right)}{\sigma \sqrt{T-t}} + \frac{1}{2} \sigma \sqrt{T-t} \right), \quad (4.69)$$

which is a bounded continuous function on $[0, T] \times (0, \infty)$. In addition, by using the alternative representation (cf. (4.59)):

$$f(t, x) = E^{P^*} \left[\left(x \exp \left(\sigma(\tilde{W}_T - \tilde{W}_t) - \frac{1}{2} \sigma^2(T-t) \right) - K^* \right)^+ \right] \quad (4.70)$$

for $(t, x) \in [0, T] \times (0, \infty)$, the uniform integrability of $\{\exp(\sigma(\tilde{W}_T - \tilde{W}_t) - \frac{1}{2} \sigma^2(T-t)), t \in [0, T]\}$, and dominated convergence, we can readily verify that f is continuous on $[0, T] \times (0, \infty)$. Note that by the continuity of f , S_t^* , and C_t^* , we may assume that P^* -a.s., $C_t^* = f(t, S_t^*)$ for all $t \in [0, T]$, (i.e., the exceptional P^* -null set does not depend on t). We can apply Itô's formula to $f(t, S_t^*)$ for $t < T$ to obtain P^* -a.s., for all $t < T$,

$$f(t, S_t^*) = f(0, S_0) + \int_0^t f_s(s, S_s^*) ds + \int_0^t f_x(s, S_s^*) dS_s^* + \frac{1}{2} \int_0^t f_{xx}(s, S_s^*) d[S^*]_s, \quad (4.71)$$

where $f_s(s, x)$ denotes the first partial derivative of $f(s, x)$ with respect to s ; $f_x(s, x)$ and $f_{xx}(s, x)$ denote the first and second partial derivatives, respectively, of $f(s, x)$ with respect to x ; and $[S^*]$ denotes the quadratic variation process associated with S^* which satisfies $d[S^*]_s = \sigma^2(S_s^*)^2 ds$ (cf. (4.30)). The first and third integrals above are path-by-path Riemann integrals and the second integral is a stochastic integral. Indeed, for each $t < T$, $\{\int_0^u f_x(s, S_s^*) dS_s^*, \mathcal{F}_u, u \in [0, t]\}$ is a continuous local martingale under P^* . On the other hand, $\{C_u^*, \mathcal{F}_u, u \in [0, t]\}$ is a continuous process and by (4.57) it is a martingale under P^* . Now, P^* -a.s., $C_u^* = f(u, S_u^*)$ for all $u \in [0, t]$, and so by (4.71) we have P^* -a.s. for all $u \in [0, t]$:

$$\int_0^u \left(f_s(s, S_s^*) + \frac{1}{2} \sigma^2(S_s^*)^2 f_{xx}(s, S_s^*) \right) ds = C_u^* - C_0^* - \int_0^u f_x(s, S_s^*) dS_s^*. \quad (4.72)$$

Now, by the above discussion, the right member above is a continuous local martingale under P^* that starts from zero, and the left member is a continuous, adapted process whose paths are of bounded variation on $[0, t]$. It follows, by the uniqueness of the decomposition of a continuous semimartingale (cf. [5], Corollary 4.5), that P^* -a.s., both sides of (4.72) are zero for all $u \in [0, t]$. Since $t < T$ was arbitrary, this in fact holds true for all $u \in [0, T)$. Indeed, not only is the left member zero P^* -a.s., but one can show (for example by direct verification using the formula for f) that f satisfies the *Black-Scholes partial*

differential equation:

$$f_t + \frac{1}{2}\sigma^2 x^2 f_{xx} = 0 \quad \text{for all } (t, x) \in [0, T) \times (0, \infty).$$

Remark. There are several different forms of the Black-Scholes partial differential equation, depending on whether one uses the function $f(t, x)$ for representing C^* or one uses the undiscounted function $e^{rt}f(t, x)$ for representing C .

Using the fact that P^* -a.s., the right member of (4.72) is zero for all $u \in [0, T)$, we have that P^* -a.s.,

$$C_t^* = C_0 + \int_0^t f_x(s, S_s^*) dS_s^* \quad \text{for all } t < T. \quad (4.73)$$

(Note that $C_0 = C_0^*$ for this.) Comparing this with the self-financing property of trading strategies and keeping in mind that we want C^* to be the discounted value process for a replicating strategy, the above suggests that we define

$$\alpha_t = f_x(t, S_t^*) \quad \text{for each } t \in [0, T).$$

The definition of α_T is not critical (since it is only in force for an instant of time), as long as it is part of a self-financing strategy. However, we can define it in such a way that $\alpha_T = \lim_{t \rightarrow T} \alpha_t$ P^* -a.s. Note that since $P^*(S_T^* = K^*) = 0$ and S^* has continuous paths, we have that P^* -a.s.,

$$\lim_{t \rightarrow T} f_x(t, S_t^*) = \lim_{t \rightarrow T} \Phi \left(\frac{\log \left(\frac{S_t^*}{K^*} \right)}{\sigma \sqrt{T-t}} + \frac{1}{2} \sigma \sqrt{T-t} \right) \quad (4.74)$$

$$= 1_{\{S_T^* > K^*\}}. \quad (4.75)$$

Thus, we may and do define $\alpha_T = 1_{\{S_T^* > K^*\}}$. In order to maintain the appropriate value process, we define

$$\beta_t = C_t^* - \alpha_t S_t^*, \quad t \in [0, T].$$

It is straightforward to verify that $\phi = \{(\alpha_t, \beta_t), t \in [0, T]\}$ is a trading strategy with discounted value process C^* . By the continuity of C^* and of the stochastic integral process $\{\int_0^t \alpha_s dS_s^*, t \in [0, T]\}$, it follows on taking the limit as $t \rightarrow T$ that (4.73) also holds at $t = T$. Thus, we have P^* -a.s., for each $t \in [0, T]$,

$$V_t^*(\phi) = \alpha_t S_t^* + \beta_t = C_t^* = C_0 + \int_0^t \alpha_s dS_s^*,$$

which implies that ϕ is self-financing. Moreover, since $\{C_t^*, \mathcal{F}_t, t \in [0, T]\}$ is a martingale under P^* , ϕ is an admissible strategy. Finally, since $V_T(\phi) = C_T = (S_T - K)^+$, ϕ is a replicating strategy for the European call option with strike price K and expiration time T . The results of this section are summarized in the following theorem.

Theorem 4.5.1 *For each $t \in [0, T)$, let*

$$C_t^* = S_t^* \Phi \left(\frac{\log \left(\frac{S_t^*}{K^*} \right)}{\sigma \sqrt{T-t}} + \frac{1}{2} \sigma \sqrt{T-t} \right) - K^* \Phi \left(\frac{\log \left(\frac{S_t^*}{K^*} \right)}{\sigma \sqrt{T-t}} - \frac{1}{2} \sigma \sqrt{T-t} \right), \quad (4.76)$$

and let $C_T^ = (S_T^* - K^*)^+$. Define $\alpha_T = 1_{\{S_T^* > K^*\}}$, and let*

$$\alpha_t = \Phi \left(\frac{\log \left(\frac{S_t^*}{K^*} \right)}{\sigma \sqrt{T-t}} + \frac{1}{2} \sigma \sqrt{T-t} \right), \quad \text{for each } t \in [0, T), \quad (4.77)$$

$$\beta_t = C_t^* - \alpha_t S_t^* \quad \text{for each } t \in [0, T]. \quad (4.78)$$

Then $\phi = \{(\alpha_t, \beta_t), t \in [0, T]\}$ is a replicating strategy for the European call option with strike price K and expiration time T , and the arbitrage free price process for this option is given by $\{e^{rt} C_t^, t \in [0, T]\}$.*