

Chapter 4

Black-Scholes Model

In this chapter we consider a simple continuous (in both time and space) financial market model called the Black-Scholes model. This can be viewed as a continuous analogue of the binomial model. Indeed, it can be obtained as a limit of a sequence of binomial models (with appropriate parameters). Here $\log$ denotes the natural logarithm.

4.1 Black-Scholes Model

We consider a finite time interval $[0, T]$ for some $T < \infty$ as the interval during which trading may take place. The Borel σ -algebra of subsets of $[0, T]$ will be denoted by $\mathcal{B}_T$.

We assume as given a complete probability space $(\Omega, \mathcal{F}, P)$ on which is defined a standard one-dimensional Brownian motion $W = \{W_t : t \in [0, T]\}$. In particular, the collection of random variables $\{W_t : t \in [0, T]\}$ satisfies

- (i) $W_0 = 0$ P -a.s.,
- (ii) $t \rightarrow W_t(\omega)$ is continuous for each $\omega \in \Omega$,
- (iii) W has independent increments, i.e., $W_{t_1} - W_{t_0}, W_{t_2} - W_{t_1}, \dots, W_{t_n} - W_{t_{n-1}}$, are independent for any $0 \leq t_0 < t_1 < \dots < t_n < \infty$ and $n = 1, 2, \dots$,
- (iv) for any $0 \leq s \leq t$, $W_t - W_s$ is normally distributed with mean zero and variance $t - s$.

For each $t \in [0, T]$, let $\mathcal{F}_t^0 = \sigma\{W_s : 0 \leq s \leq t\}$, the smallest σ -algebra with respect to which W_s is measurable for each $0 \leq s \leq t$. A P -null set is a set in $\mathcal{F}$ of probability zero under P . For each $t \in [0, T]$, let $\mathcal{F}_t$ denote the smallest

σ -algebra containing $\mathcal{F}_t^0$ and all of the P -null sets in $\mathcal{F}$. Then $\{\mathcal{F}_t : t \in [0, T]\}$ is called the filtration generated by the Brownian motion W under P . It is well known that this filtration is right continuous, i.e., for each $t \in [0, T)$, $\mathcal{F}_t = \mathcal{F}_{t+} \equiv \bigcap_{s \in (t, T]} \mathcal{F}_s$, cf. Chung [4], Section 2.3, Theorem 4. All random variables considered in this chapter will be assumed to be defined on $(\Omega, \mathcal{F}_T)$, and so without loss of generality we assume that $\mathcal{F} = \mathcal{F}_T$. Expectations with respect to P will be denoted by $E[\cdot]$, unless there is more than one probability measure under consideration, in which case we shall use E^P in place of E .

A (stochastic) process is a collection of real-valued random variables $\{Z_t, t \in [0, T]\}$ defined on $(\Omega, \mathcal{F}, P)$. Such a process is adapted if Z_t is $\mathcal{F}_t$ -measurable for each $t \in [0, T]$. The process is (right) continuous if its paths are (right) continuous. Two stochastic processes $\{Y_t, t \in [0, T]\}$ and $\{Z_t, t \in [0, T]\}$ defined on $(\Omega, \mathcal{F}, P)$ are

- (i) *versions of one another* if $P(Y_t = Z_t) = 1$ for all $t \in [0, T]$,
- (ii) *indistinguishable* if $P(Y_t = Z_t \text{ for all } t \in [0, T]) = 1$.

(Note that for (ii) it is implicit that the event $\{Y_t = Z_t \text{ for all } t \in [0, T]\}$ is measurable.) Two right continuous processes that are versions of one another are indistinguishable. Two processes will be regarded as being equal if they are indistinguishable.

A *martingale* is a stochastic process $M = \{M_t, t \in [0, T]\}$ such that

- (i) $E[|M_t|] < \infty$,
- (ii) M is adapted, i.e., $M_t \in \mathcal{F}_t$ for each $t \in [0, T]$,
- (iii) $E[M_t | \mathcal{F}_s] = M_s$ for all $s \in [0, t]$.

A martingale M is an L^2 -martingale if $E[|M_t|^2] < \infty$ for each $t \in [0, T]$.

Our Black-Scholes model has two assets, a (risky) stock with price process $S = \{S_t, t \in [0, T]\}$ and a (riskless) bond with price process $B = \{B_t, t \in [0, T]\}$. These processes are given by

$$S_t = S_0 \exp \left(\left(\mu - \frac{1}{2} \sigma^2 \right) t + \sigma W_t \right), \quad t \in [0, T], \quad (4.1)$$

$$B_t = e^{rt} \quad t \in [0, T], \quad (4.2)$$

where $r > 0$, $\mu \in \mathbb{R}$, $\sigma > 0$, and $S_0 > 0$ is a positive constant. The parameter σ is called the *volatility*. It is well known (cf. Chung and Williams [5], Theorem 6.2) that $\{e^{-\mu t} S_t, \mathcal{F}_t, t \in [0, T]\}$ is an L^2 -martingale under P . The processes S, B are continuous, adapted, and satisfy the following dynamic equations under P :

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \quad t \in [0, T], \quad (4.3)$$

$$dB_t = r B_t dt, \quad t \in [0, T], \quad (4.4)$$

Remark. To see the connection with the binomial model, note that in the Black-Scholes model, $\log(S_t/S_0) = (\mu - \frac{1}{2}\sigma^2)t + \sigma W_t$, a Brownian motion with drift, whereas in the binomial model, $\log(S_t/S_0) = \sum_{i=1}^t \xi_i$, which is a (possibly biased) random walk. It is well known that a random walk with appropriate rescaling and parameter values can be approximated by a Brownian motion with drift.

A *trading strategy* is a two-dimensional stochastic process $\phi = \{\phi_t = (\alpha_t, \beta_t), t \in [0, T]\}$ satisfying

- (i) $\phi : [0, T] \times \Omega \rightarrow \mathbb{R}^2$ is $(\mathcal{B}_T \times \mathcal{F}_T)$ -measurable, where $\phi(t, \omega) = \phi_t(\omega)$,
- (ii) ϕ is adapted, i.e., $\phi_t \in \mathcal{F}_t$ for each $t \in [0, T]$,
- (iii) $\int_0^T \alpha_t^2 dt < \infty$ and $\int_0^T |\beta_t| dt < \infty$ P -a.s.

These conditions ensure that integrals of the form $\int_0^t \alpha_s dS_s$ and $\int_0^t \beta_s dB_s$ are finite P -a.s. and when considered as functions of t define adapted stochastic processes. Here α_t is interpreted as the number of shares of stock held at time t and β_t is interpreted as the number of bonds held at time t . The *value* at time t of the portfolio associated with ϕ is given by

$$V_t(\phi) = \alpha_t S_t + \beta_t B_t. \quad (4.5)$$

A trading strategy ϕ is said to be *self-financing* if P -a.s.,

$$V_t(\phi) = V_0(\phi) + \int_0^t \alpha_s dS_s + \int_0^t \beta_s dB_s, \quad t \in [0, T], \quad (4.6)$$

or, in other words,

$$dV_t(\phi) = \alpha_t dS_t + \beta_t dB_t, \quad t \in [0, T]. \quad (4.7)$$

Thus, changes in the value of the portfolio result only from changes in the values of the assets, so that there is no external infusion of capital and no spending of wealth.

Let ϕ be a trading strategy. The discounted stock price process and discounted value process associated with ϕ are given by

$$S_t^* = \frac{S_t}{B_t} = e^{-rt} S_t, \quad t \in [0, T], \quad (4.8)$$

$$V_t^*(\phi) = \frac{V_t(\phi)}{B_t} = e^{-rt} V_t(\phi), \quad t \in [0, T], \quad (4.9)$$

respectively. In particular, by (4.3) we have under P :

$$dS_t^* = -rS_t^* dt + e^{-rt} dS_t = (\mu - r)S_t^* dt + \sigma S_t^* dW_t, \quad t \in [0, T]. \quad (4.10)$$

Lemma 4.1.1 *A trading strategy ϕ is self-financing if and only if P -a.s.,*

$$V_t^*(\phi) = V_0(\phi) + \int_0^t \alpha_s dS_s^*, \quad \text{for all } t \in [0, T]. \quad (4.11)$$

PROOF. In the following, for convenience, we use the differential forms of integral equations which hold P -a.s. However, each of the steps could be written in integral form, to give a totally rigorous proof. The essential aspect is that we use the rules of stochastic calculus for manipulating differentials at each stage.

Suppose that ϕ is a self-financing trading strategy. Then, by (4.9), (4.5), and (4.7), we have for each $t \in [0, T]$,

$$\begin{aligned} dV_t^*(\phi) &= -re^{-rt}V_t(\phi)dt + e^{-rt}dV_t(\phi) \\ &= e^{-rt}(-r\alpha_t S_t dt - r\beta_t B_t dt + \alpha_t dS_t + \beta_t dB_t). \end{aligned}$$

Using (4.3), (4.4), and (4.10), the above yields

$$dV_t^*(\phi) = e^{-rt}\alpha_t((\mu - r)S_t dt + \sigma S_t dW_t) = \alpha_t dS_t^*, \quad (4.12)$$

for each $t \in [0, T]$, which proves that (4.11) holds.

Conversely, suppose that (4.11) holds. Then using (4.10), (4.4), and (4.5), for each $t \in [0, T]$ we have

$$\begin{aligned} dV_t^*(\phi) &= \alpha_t dS_t^* = e^{-rt}\alpha_t(-rS_t dt + dS_t) \\ &= e^{-rt}(-r\alpha_t S_t dt - r\beta_t B_t dt + \beta_t dB_t + \alpha_t dS_t) \\ &= e^{-rt}(-rV_t(\phi) dt + \alpha_t dS_t + \beta_t dB_t). \end{aligned}$$

Then,

$$\begin{aligned} dV_t(\phi) &= d(e^{rt}V_t^*(\phi)) = re^{rt}V_t^*(\phi)dt + e^{rt}dV_t^*(\phi) \\ &= rV_t(\phi)dt + (-rV_t(\phi)dt + \alpha_t dS_t + \beta_t dB_t) \\ &= \alpha_t dS_t + \beta_t dB_t, \end{aligned}$$

and hence ϕ is self-financing. $\square$

A self-financing trading strategy ϕ is an *arbitrage opportunity* if

$$V_0(\phi) = 0, \quad V_T(\phi) \geq 0, \quad \text{and} \quad E[V_T(\phi)] > 0. \quad (4.13)$$

In contrast to the discrete time case, where we had only finitely many trading times and gains or losses are controlled by the initial investment, in continuous time, one can change one's strategy infinitely many times in a finite time interval and thereby obtain unbounded gains or losses by use of a doubling type of strategy. The following is a concrete example of such a strategy.

Example. Consider the Black-Scholes model with $r = 0$, $\mu = 0$, and $\sigma = 1$. Then

$$B_t = 1 \quad \text{and} \quad dS_t = S_t dW_t, \quad t \in [0, T]. \quad (4.14)$$

Define

$$I(t) = \int_0^t \frac{1}{\sqrt{T-s}} dW_s, \quad \text{for all } 0 \leq t < T. \quad (4.15)$$

Then the quadratic variation of I is given by

$$[I]_t = \int_0^t \frac{1}{T-s} ds = \log \left(\frac{T}{T-t} \right), \quad \text{for each } t < T. \quad (4.16)$$

Note that $[I]_0 = 0$, $[I]_t$ is increasing with t , and $[I]_t \rightarrow \infty$ as $t \rightarrow T$. It follows that I can be time changed to a Brownian motion (cf. Chung-Williams [5], Section 9.3) and in particular, for each $a > 0$ and

$$\tau_a \equiv \inf\{t \in [0, T] : I(t) = a\} \wedge T, \quad (4.17)$$

we have $0 < \tau_a < T$ P -a.s. Fix $a > 0$ and let $\phi = \{(\alpha_t, \beta_t), t \in [0, T]\}$, where for each $t \in [0, T]$,

$$\alpha_t = (T-t)^{-\frac{1}{2}} S_t^{-1} 1_{\{t \leq \tau_a\}}, \quad (4.18)$$

$$\beta_t = I(t \wedge \tau_a) - \alpha_t S_t. \quad (4.19)$$

Then the value process for ϕ is given by

$$V_t(\phi) = I(t \wedge \tau_a) = \int_0^{t \wedge \tau_a} \frac{1}{\sqrt{T-s}} dW_s \quad (4.20)$$

$$= \int_0^t \alpha_s S_s dW_s = \int_0^t \alpha_s dS_s, \quad (4.21)$$

where we have used the fact that S solves (4.3) with $\mu = 0$ and $\sigma = 1$. Since $r = 0$, it follows that (4.11) holds and hence ϕ is a self-financing trading strategy. Note that $V_0(\phi) = 0$ and $V_T(\phi) = I(\tau_a) = a > 0$ and so ϕ is an arbitrage opportunity.

There are several ways to add conditions to ϕ to rule out such arbitrage strategies. These usually amount to constraints on the size of integrals of ϕ with respect to $d(S, B)$. Since pricing of contingent claims based on this model will use an equivalent martingale measure (or risk neutral probability), these conditions are often phrased in terms of such a probability measure. Some authors require that the associated value process is L^2 -bounded under an equivalent martingale measure, some require that the discounted value process is a martingale under such a measure, and some put integrability constraints directly on ϕ . Before discussing our conditions on ϕ , we introduce the equivalent martingale measure.

4.2 Equivalent Martingale Measure

Definition 4.2.1 Two probability measures Q and $\tilde{Q}$ on $(\Omega, \mathcal{F})$ are equivalent (or mutually absolutely continuous) if

$$Q(A) = 0 \quad \text{is equivalent to} \quad \tilde{Q}(A) = 0 \quad \text{for each } A \in \mathcal{F}. \quad (4.22)$$

An equivalent martingale measure, also called a risk neutral probability, is a probability measure P^* on $(\Omega, \mathcal{F})$ such that P^* is equivalent to P and $\{S_t^*, \mathcal{F}_t, t \in [0, T]\}$ is a martingale under P^* .

We wish to show that such a P^* exists. For this note that

$$S_t^* = S_0 \exp \left(\left(\mu - r - \frac{1}{2} \sigma^2 \right) t + \sigma W_t \right) \quad (4.23)$$

and in particular, P -a.s.,

$$dS_t^* = (\mu - r) S_t^* dt + \sigma S_t^* dW_t, \quad t \in [0, T]. \quad (4.24)$$

The second term on the right in (4.24), when integrated, defines a martingale under P . We seek a probability measure P^* equivalent to P such that the discounted stock price process S^* satisfies a differential equation like (4.24), but without the drift term $(\mu - r) S_t^* dt$. For this, we use a simple form of the Girsanov transformation for changing the drift of a Brownian motion. Let

$$\theta = \frac{\mu - r}{\sigma},$$

$$\Lambda_t = \exp \left(-\theta W_t - \frac{1}{2} \theta^2 t \right), \quad t \in [0, T].$$

Then it is well known (cf. [5], Theorem 6.2) that $\{\Lambda_t, \mathcal{F}_t, t \in [0, T]\}$ is a positive martingale under P . On $(\Omega, \mathcal{F})$, we define a new probability measure P^* so that

$$\frac{dP^*}{dP} = \Lambda_T \quad \text{on } \mathcal{F}, \quad (4.25)$$

i.e.,

$$P^*(A) = E^P [1_A \Lambda_T], \quad A \in \mathcal{F}, \quad (4.26)$$

where we have added the superscript P to the expectation E to emphasize that it is computed under the probability measure P . Similarly, we shall use E^{P^*} to denote the expectation operator under P^* . Note that, since $\{\Lambda_t, \mathcal{F}_t, t \in [0, T]\}$ is a P -martingale, $P^*(\Omega) = E^P [\Lambda_T] = \Lambda_0 = 1$ and so P^* is indeed a probability measure on $(\Omega, \mathcal{F})$. Since $\Lambda_T > 0$, it follows that for each $A \in \mathcal{F}$, $P^*(A) = 0$ if and only if $P(A) = 0$. Thus, P^* is equivalent to P . All that remains is to show that the discounted stock price process S^* is a martingale under P^* . For this, let

$$\tilde{W}_t = W_t + \theta t, \quad t \in [0, T]. \quad (4.27)$$

By Girsanov's theorem (cf. [5], Section 9.4), $\{\tilde{W}_t, \mathcal{F}_t, t \in [0, T]\}$ is a standard Brownian motion under P^* . Thus, for each $t \in [0, T]$, by (4.23) we have

$$S_t^* = S_0 \exp \left(\left(\mu - r - \frac{1}{2} \sigma^2 \right) t + \sigma (\tilde{W}_t - \theta t) \right) \quad (4.28)$$

$$= S_0 \exp \left(-\frac{1}{2} \sigma^2 t + \sigma \tilde{W}_t \right), \quad (4.29)$$

and so P^* -a.s.,

$$dS_t^* = \sigma S_t^* d\tilde{W}_t, \quad t \in [0, T], \quad (4.30)$$

where $\tilde{W}$ is a standard Brownian motion under P^* . It is well known that the form (4.29) is that of an L^2 -martingale with respect to $\{\mathcal{F}_t\}$, under P^* (cf. [5], Theorem 6.2). Thus, P^* is an equivalent martingale measure. In fact, P^* is the unique equivalent martingale measure.

An *admissible strategy* is a self-financing trading strategy ϕ such that $\{V_t^*(\phi), \mathcal{F}_t, t \in [0, T]\}$ is a martingale under the equivalent martingale measure P^* .

4.3 European Contingent Claims

A *European contingent claim* is represented by an $\mathcal{F}_T$ -measurable random variable X . A *replicating strategy* for a European contingent claim $X \in \mathcal{F}_T$ is an admissible strategy ϕ such that $V_T(\phi) = X$.

We shall use the following form of the martingale representation theorem for Brownian motion to obtain a replicating strategy for suitably integrable X . For this theorem, it is important that we are using the filtration generated by the Brownian motion W (cf. Section 4.1).

A random variable $\tau : \Omega \rightarrow [0, T] \cup \{+\infty\}$ is a *stopping time* if

$$\{\omega \in \Omega : \tau(\omega) \leq t\} \in \mathcal{F}_t \quad \text{for each } t \in [0, T]. \quad (4.31)$$

A real-valued process $M = \{M_t, t \in [0, T]\}$ is a *local martingale* under P^* if it is right continuous, adapted, and there exists an increasing sequence of stopping times $\{\tau_n : n = 1, 2, \dots\}$ taking values in $[0, T] \cup \{+\infty\}$ such that

$$P^*(\tau_n \geq T \text{ for all sufficiently large } n) = 1, \quad (4.32)$$

and for each n , $M^n = \{M_{t \wedge \tau_n}, \mathcal{F}_t, t \in [0, T]\}$ is a martingale under P^* .

Theorem 4.3.1 *Suppose that $M = \{M_t, \mathcal{F}_t, t \in [0, T]\}$ is a local martingale under P^* . Then there exists a $(\mathcal{B}_T \times \mathcal{F}_T)$ -measurable, adapted process $\eta = \{\eta_t : t \in [0, T]\}$ such that P^* -a.s., $\int_0^T \eta_s^2 ds < \infty$ and*

$$M_t = M_0 + \int_0^t \eta_s d\tilde{W}_s, \quad \text{for all } t \in [0, T]. \quad (4.33)$$

In particular, M has a continuous version.

PROOF. For a proof of this martingale representation theorem, see Revuz and Yor [13], Theorem V.3.5. $\square$

Remark. Note that P and P^* have the same null sets and so an equality holds P -a.s. if and only if it holds P^* -a.s. By (4.27) and the equivalence of P^* and P , we have that $\mathcal{F}_t = \sigma\{\tilde{W}_s, s \in [0, t]\}^\sim$ for each $t \in [0, T]$, where the superscript $\sim$ denotes augmentation by the P^* -null sets.

Theorem 4.3.2 *Suppose that X is an $\mathcal{F}_T$ -measurable random variable such that $E^{P^*}[|X|] < \infty$. Then there exists a replicating strategy ϕ for X . Moreover, for $X^* = X/B_T$, we have that for each $t \in [0, T]$, P^* -a.s.,*

$$V_t^*(\phi) = E^{P^*}[X^* | \mathcal{F}_t]. \quad (4.34)$$

PROOF. We first prove that (4.34) holds if ϕ is a replicating strategy for X . Indeed, for such a ϕ , we have for fixed $t \in [0, T]$ that P^* -a.s.,

$$V_t^*(\phi) = E^{P^*}[V_T^*(\phi) | \mathcal{F}_t] = E^{P^*}[X^* | \mathcal{F}_t], \quad (4.35)$$

where the first equality follows from the admissibility of ϕ which implies that $\{V_t^*(\phi), \mathcal{F}_t, t \in [0, T]\}$ is a martingale under P^* , and the second equality follows from the replicating property of ϕ : $V_T(\phi) = X$.

In light of the above, to prove the existence of a replicating strategy ϕ , it is natural to consider

$$M_t = E^{P^*}[X^* | \mathcal{F}_t], \quad t \in [0, T]. \quad (4.36)$$

Note that $\{M_t, \mathcal{F}_t, t \in [0, T]\}$ is a P^* -martingale and hence it has a right continuous version. In fact, since the filtration is generated by Brownian motion, M has a continuous version (cf. Revuz and Yor [13], Theorems II.2.9, V.3.5). We again denote such a continuous version by M . Thus, M is a local martingale (take $\tau_n = +\infty$ for all n in the definition) and so by Theorem 4.3.1, there is an adapted process $\eta = \{\eta_t, t \in [0, T]\}$ satisfying the conditions in Theorem 4.3.1 and such that for each $t \in [0, T]$, P^* -a.s.,

$$M_t = M_0 + \int_0^t \eta_s d\tilde{W}_s. \quad (4.37)$$

Define

$$\alpha_t = \frac{\eta_t}{\sigma S_t^*}, \quad t \in [0, T]. \quad (4.38)$$

Then $\alpha = \{\alpha_t, t \in [0, T]\}$ is $(\mathcal{B}_T \times \mathcal{F}_T)$ -measurable, adapted and

$$\int_0^T \alpha_t^2 dt \leq \left(\sigma \inf_{t \in [0, T]} S_t^* \right)^{-2} \int_0^T \eta_t^2 dt. \quad (4.39)$$

Note that $S_t^* > 0$ for each $t \in [0, T]$, and S_t^* is a continuous function of $t \in [0, T]$. Therefore, $\inf_{t \in [0, T]} S_t^* > 0$. So it follows from the integrability property of η that P^* -a.s.,

$$\int_0^T \alpha_t^2 dt < \infty. \quad (4.40)$$

Define

$$\beta_t = M_t - \alpha_t S_t^*, \quad t \in [0, T]. \quad (4.41)$$

Then $\beta = \{\beta_t, t \in [0, T]\}$ is $(\mathcal{B}_T \times \mathcal{F}_T)$ -measurable, adapted and using Cauchy's inequality we have

$$\int_0^T |\beta_t| dt \leq \int_0^T |M_t| dt + \int_0^T |\alpha_t| S_t^* dt \quad (4.42)$$

$$\leq T \max_{t \in [0, T]} |M_t| + \left(\max_{t \in [0, T]} S_t^* \right) \left(\int_0^T \alpha_t^2 dt \right)^{\frac{1}{2}} T^{\frac{1}{2}}. \quad (4.43)$$

The last line above is finite P^* -a.s., by (4.40) and the fact that M and S^* are continuous processes. Let $\phi = \{(\alpha_t, \beta_t) : t \in [0, T]\}$. Then,

$$V_t^*(\phi) = \alpha_t S_t^* + \beta_t = M_t, \quad t \in [0, T]. \quad (4.44)$$

This, together with (4.33), the definition of α , and (4.30), gives P^* -a.s.,

$$V_t^*(\phi) = V_0(\phi) + \int_0^t \eta_s d\tilde{W}_s \quad (4.45)$$

$$= V_0(\phi) + \int_0^t \alpha_s \sigma S_s^* d\tilde{W}_s \quad (4.46)$$

$$= V_0(\phi) + \int_0^t \alpha_s dS_s^*, \quad (4.47)$$

for each $t \in [0, T]$. Therefore, since P and P^* have the same null sets, ϕ is a self-financing trading strategy. Combining this with (4.44) and the martingale property of M , we see that ϕ is admissible. Finally, taking $t = T$ in (4.44) implies that ϕ replicates X since $M_T = X^*$. $\square$