

Let B denote the closed ball centered at the origin of radius $r > 0$ (where distance is measured by the Euclidean norm). Choose $r > 0$ such that $B \cap G \neq \emptyset$. Then $B \cap G$ is closed, bounded, non-empty and hence compact. So the continuous function $g(x) = \|x\|$ attains its infimum on $B \cap G$ at some $z \in B \cap G$, where $\|x\| = (x \cdot x)^{\frac{1}{2}}$ denotes the Euclidean norm of x . Note that $\|z\| \leq r$, since $z \in B$.

Now $\|x\| > r$ for $x \in G \setminus B$ and so combining this with the above we have $\|x\| \geq \|z\|$ for all $x \in G$. Then for any $\lambda \in (0, 1)$ and $x \in G$, $\lambda x + (1 - \lambda)z \in G$ by the convexity of G and so

$$\|\lambda x + (1 - \lambda)z\|^2 \geq \|z\|^2 \text{ for all } x \in G, \lambda \in (0, 1).$$

Expanding and dividing through by λ yields:

$$2(1 - \lambda)x \cdot z - 2z \cdot z + \lambda(x \cdot x + z \cdot z) \geq 0.$$

Letting $\lambda \rightarrow 0$, we obtain

$$x \cdot z \geq z \cdot z \quad \text{for all } x \in G.$$

Then, $(f - \ell) \cdot z \geq z \cdot z$ for all $f \in F$, $\ell \in L$, which implies

$$f \cdot z \geq \ell \cdot z + z \cdot z \quad \text{for all } f \in F, \ell \in L.$$

Fix $f \in F$. Then

$$\ell \cdot z \leq f \cdot z - z \cdot z \quad \text{for all } \ell \in L.$$

But L is a linear space, so the above holds with $\gamma\ell$ in place of ℓ for all $\gamma \in \mathbb{R}$. The only way this can be true is if $\ell \cdot z = 0$ for all $\ell \in L$. Hence, $f \cdot z \geq z \cdot z > 0$ for all $f \in F$.

Let $H = \{x \in \mathbb{R}^n : x \cdot z = 0\}$. Then from the above, $L \subset H$ and $f \cdot z > 0$ for all $f \in F$. $\square$

3.7 Exercises

1. Consider the multi-period CRR binomial model introduced in Chapter 2. Assuming $u > d > 0$, verify that this model is viable if and only if $d < 1 + r < u$. In this case, verify that the model is complete.

2. Let $T = 2$, $\Omega = \{\omega_1, \dots, \omega_4\}$, $P(\{\omega_i\}) > 0$ for $i = 1, \dots, 4$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1 = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, \emptyset, \Omega\}$, and $\mathcal{F}_2$ be the collection of all subsets of Ω . Consider a riskless asset with price process $S^0 = \{S_t^0, t = 0, 1, 2\}$ where $S_t^0 = 1$ for all t , and a risky asset with price process $S^1 = \{S_t^1, t = 0, 1, 2\}$ such that

$$S_0^1(\omega_1) = 5, \quad S_1^1(\omega_1) = 8, \quad S_2^1(\omega_1) = 9 \quad (3.72)$$

$$S_0^1(\omega_2) = 5, \quad S_1^1(\omega_2) = 8, \quad S_2^1(\omega_2) = 6 \quad (3.73)$$

$$S_0^1(\omega_3) = 5, \quad S_1^1(\omega_3) = 4, \quad S_2^1(\omega_3) = 6 \quad (3.74)$$

$$S_0^1(\omega_4) = 5, \quad S_1^1(\omega_4) = 4, \quad S_2^1(\omega_4) = 3. \quad (3.75)$$

Then

$$X = \max(0, S_0^1 - 7, S_1^1 - 7, S_2^1 - 7), \quad (3.76)$$

is the value at time T of a so-called *look-back* option, where this value depends on the prices of the underlying asset S^1 in the past as well as at time T .

- (a) Draw a tree to indicate the possible “paths” followed by the risky asset price process S^1 .
- (b) Find an equivalent martingale measure for the model.
- (c) Find a replicating strategy for the option whose value at time T is given by X .
- (d) What is the arbitrage free price for the option at time zero?

3. Consider a finite market model with $T = 2$, $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}$, and $P(\{\omega_i\}) > 0$ for $i = 1, \dots, 5$. Suppose there are two assets, a riskless asset with price process $S^0 = \{S_t^0, t = 0, 1, 2\}$ where $S_t^0 = (1+r)^t$ for $t = 0, 1, 2$, and some $r \geq 0$, and a risky asset with price process $S^1 = \{S_t^1, t = 0, 1, 2\}$ where

$$S_0^1(\omega_1) = 5, \quad S_1^1(\omega_1) = 8, \quad S_2^1(\omega_1) = 9 \quad (3.77)$$

$$S_0^1(\omega_2) = 5, \quad S_1^1(\omega_2) = 8, \quad S_2^1(\omega_2) = 7 \quad (3.78)$$

$$S_0^1(\omega_3) = 5, \quad S_1^1(\omega_3) = 8, \quad S_2^1(\omega_3) = 6 \quad (3.79)$$

$$S_0^1(\omega_4) = 5, \quad S_1^1(\omega_4) = 4, \quad S_2^1(\omega_4) = 6 \quad (3.80)$$

$$S_0^1(\omega_5) = 5, \quad S_1^1(\omega_5) = 4, \quad S_2^1(\omega_5) = 3. \quad (3.81)$$

Let $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1 = \sigma\{S_0^1, S_1^1\}$ and $\mathcal{F}_2 = \sigma\{S_0^1, S_1^1, S_2^1\}$.

- (a) Draw a tree to indicate the possible “paths” followed by the risky asset price process S^1 .
- (b) Suppose $r = 0.1$. Is there an equivalent martingale measure for this model? If there is one, is it unique? What are the answers to the last two questions if $r = 1$?

4. Let $T = 1$, $S_t^0 = (1+r)^t$ where $r = \frac{1}{9}$, $t = 0, 1$, $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$. Suppose that there are two risky assets with price processes S^1, S^2 :

i	S_0^i	S_1^i			
		ω_1	ω_2	ω_3	ω_4
1	5	20/3	20/3	40/9	20/9
2	10	40/3	80/9	80/9	40/3

- (i) Specify the space

$$L = \{G_T^*(\phi) : \phi \text{ is a self-financing trading strategy with } V_0(\phi) = 0\}$$

for this example.

- (ii) Are there any arbitrage opportunities for this example? If so, find all of them.
- (iii) Are there any equivalent martingale measures for this example? If so, find all of them.

5. Show that the separating hyperplane theorem follows from the Hahn-Banach theorem.