

2.4 Exercises

1. Consider a single period CRR model with $S_0 = \$50$, $S_1 = \$100$ or $\$25$, $r = 0.25$.

- (a) Find the arbitrage free initial price of a European call option for one share of stock where the strike price is $K = \$50$ and the exercise time $T = 1$.
- (b) Find a hedging strategy that replicates the value of the option described in (a).
- (c) Suppose the option in (a) is initially priced at \$1 above the arbitrage free price. Describe a strategy (for trading in stock, bond and the option) that is an arbitrage.
- (d) What is the arbitrage free initial price for a put option with the same strike price and exercise time as the call option described in (a)?

2. Consider a CRR model with $T = 2$, $S_0 = \$50$, $S_1 = \$100$ or $S_1 = \$25$, and an associated European call option with strike price $K = \$40$ and exercise time $T = 2$. Assume that the risk free interest rate is $r = 0.1$.

- (a) Draw the binary tree and compute the arbitrage free initial price of the European call option at time zero.
- (b) Determine an explicit hedging strategy for this option.
- (c) Suppose that the option in (a) is initially priced \$2 below the arbitrage free price. Describe a strategy (for trading in stock, bond and the option) that is an arbitrage.
- (d) If you are ambitious, try to automate the pricing of a European call option in a computer program where T, S_0, u, d, K are variables.

3. Consider the multi-period CRR model and a European contingent claim X . Suppose that trading in the contingent claim is allowed at each of the times $t = 0, 1, \dots, T-1$, where the price of the contingent claim at time t is given by an $\mathcal{F}_t$ -measurable random variable C_t . Let ϕ^* be a replicating strategy (in stock and bond) for the contingent claim X . Show that

$$V_t(\phi^*) = \frac{1}{(1+r)^{T-t}} E^{p^*}[X | \mathcal{F}_t], \quad t = 0, 1, \dots, T. \quad (2.74)$$

Formulate a notion of an arbitrage opportunity in the market consisting of stock, bond and contingent claim. Show that the process $C_t = V_t(\phi^*)$, $t = 0, 1, \dots, T-1$ defines the unique arbitrage free price process for the contingent claim.

4. Consider a CRR model with terminal time T . A European call option with strike price K and expiration date T has value $C_T = (S_T - K)^+$ at time T . A European put option with the same strike price and expiration date has value $P_T = (K - S_T)^+$ at time T . Note that $C_T - P_T = S_T - K$. Let $\{C_t, t = 0, 1, \dots, T-1\}$ and $\{P_t, t = 0, 1, \dots, T-1\}$ denote the arbitrage free price processes for the call and put options, respectively (cf. Exercise 3). Show that the following call-put parity relation holds:

$$C_t - P_t = S_t - K/(1+r)^{T-t}, \quad t = 0, 1, \dots, T-1. \quad (2.75)$$

Note, you can show this either using a “no arbitrage” argument or using the formulas for C_t, P_t resulting from Exercise 3.

5. Consider a CRR model with $T = 2$, $S_0 = \$50$, $S_1 = \$100$ or $S_1 = \$25$ (the same model as in Exercise 2). Now consider an American put option with strike price $K = \$40$. Assume that the risk free interest rate is $r = 0.1$.

- (a) Use a binary tree to compute the arbitrage free initial price of the American put option.
- (b) Determine an explicit perfect hedging strategy ϕ^* for this option.
- (c) Suppose that you can buy the American put option at time zero for \$1 less than its arbitrage free price. Explicitly describe a strategy that yields an arbitrage opportunity for a buyer of the American put option.
- (d) If you are ambitious, try to automate the arbitrage free pricing of an American put option in a computer program where T, S_0, u, d, K are variables.

6. Prove that the arbitrage free initial price for an American call option is the same as the arbitrage free initial price for a European call option. Furthermore, show that with this initial (arbitrage free) price, if the buyer of an American call option waits until the expiration date to exercise the option, this prevents the seller from making a risk free profit (that is, it prevents a seller’s arbitrage).

7. This exercise will require you to obtain historical data on the daily closing price of Intel stock. Such data can be readily obtained from the Yahoo finance web site. (A link to this site is provided from the Related Links page accessible from the bottom of the Math 294 web page.)

Consider a binomial model with $p = 0.5$, i.e., the stock price process for $t = 0, 1, \dots, T$ is given by

$$S_t = S_{t-1}\xi_t, \quad t = 1, \dots, T,$$

where S_0 is a positive constant, $\{\xi_t, t = 1, \dots, T\}$ is a sequence of i.i.d. random variables with

$$P(\xi_t = u) = p, \quad P(\xi_t = d) = 1 - p,$$

$0 < d < 1 + r < u$, and r is the risk-free interest rate. Let

$$X_t = \frac{S_t - S_{t-1}}{S_{t-1}} = \xi_t - 1, \quad t = 1, \dots, T.$$

Then, with $p = 0.5$, the mean and variance of X_t are given by

$$\mu = E^p[X_t] = pu + (1-p)d - 1 = \frac{u+d}{2} - 1, \quad (2.76)$$

$$Var = E^p[(X_t - \mu)^2] = \frac{(u-d)^2}{4}, \quad (2.77)$$

respectively.

Suppose that you have a sample of closing stock prices $\tilde{S}_0, \tilde{S}_1, \dots, \tilde{S}_T$ for times $t = 0, 1, \dots, T$. For $t = 1, \dots, T$, let

$$\tilde{X}_t = \frac{\tilde{S}_t - \tilde{S}_{t-1}}{\tilde{S}_{t-1}}.$$

Then, the usual (unbiased) statistical estimators for μ and Var are given by

$$\bar{\mu} = \frac{1}{T} \sum_{t=1}^T \tilde{X}_t, \quad \overline{Var} = \frac{1}{T-1} \sum_{t=1}^T (\tilde{X}_t - \bar{\mu})^2.$$

- (a) Compute these estimators using the daily closing stock price data for Intel (ticker symbol INTC), commencing with December 1, 2002 and ending with December 31, 2002.

You can obtain this historical data in downloadable Excel spreadsheet form from the Yahoo finance website <http://finance.yahoo.com/>. Once there, type in the ticker symbol INTC, select “Chart” and click on “Get”, at the bottom of the chart area click on “Historical Prices”, type in the range of dates for which you want data and select “Daily”, then click on “Get Data”. The historical data for the range of dates will be displayed and can be saved in spreadsheet format as a .csv file by clicking on “Download Spreadsheet Format” at the bottom of the table. Once you have saved the data to a file, you can open that file using Excel (note that it has a .csv extension, not a .xls extension). You can then use the built in Excel functions Average and Var to compute the above estimators for a column of data – you can also use Stdev instead of Var to obtain the square root of Var. Here, the natural choice for one unit of time is one trading day (ignore non-trading days such as weekends and holidays in counting time).

Using these estimates in place of μ and Var in (2.76) and (2.77) respectively, determine estimates for u and d .

- (b) Now consider the binomial model with these estimated values of u and d .
Based on this binomial model, find the arbitrage free initial price on December 31, 2002 for

- (i) a European call option based on Intel stock with a strike price of \$17.50 and an expiration month of January (2003),
- (ii) an American put option based on Intel stock with a strike price of \$17.50 and an expiration month of January (2003).

Notes:

- (N1) To estimate the daily risk-free interest rate r , use the 3-month T-bill rate on December 31, 2002 of 1.19% per annum. As an approximation, assume there are $65 = 13 \times 5$ trading days in a 3-month period, set $(1 + r)^{65} = 1 + \frac{0.0119}{4}$ and solve for r .
 - (N2) Recall that all market-traded options (such as we are considering) effectively expire on the third Friday of their month of expiry (we say effectively because options cannot be traded on a Saturday). For January 2003, the third Friday was January 17, 2003. Starting with December 31, 2002 as $t = 0$, the terminal time, January 17, 2003, will be $T = 12$. The price of Intel stock on December 31, 2002 was \$15.57.
 - (N3) To compute the initial arbitrage free prices, you may want to use an Excel spreadsheet program and possibly a computer program.
- (c) Comment on your findings, especially in comparison to the option prices listed in the Wall Street Journal for Intel corresponding to the close of trading on December 31, 2002 (see the handout from the first lecture): a call option with a strike price of \$17.50 and expiration in January 2003 was priced at \$0.30 and the put option with the same strike price and expiration month was priced at \$2.15.