# Math 285A: Lecture #9

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May 2, 2007

Today's lecture:

- Finish sketch of ergodic theorem proof.
- Reversible Markov chains.
- Introduction to hidden Markov models.

## 1 Sketch of Proof of Ergodic Theorem

Recap of proof so far: Suppose the Markov chain $X$ is reversible, positive recurrent, with $X(0) = i$. Fix a state $k$.

$$\begin{aligned} W_n^k &= \sum_{l=0}^{n} T_l^k \qquad n = 0, 1, 2, \ldots \\ &= \text{time of } (n+1)^{th} \text{ visit to state } k \\ W_{-1}^k &= 0 \end{aligned}$$

where $T_l^k$ is the interoccurrence time between $l^{th}$ and $(l+1)^{th}$ visit to state $k$. Let

$$\begin{aligned} V_k(n) &= \sum_{l=0}^{n-1} \mathbf{1}_{\{X_l = k\}} \\ &= \text{amount of time spent in } k \text{ up to time } (n-1) \\ &= \# \text{ of visits to } k \text{ up to time } (n-1) \end{aligned}$$

then we have that

$$W_{V_k(n)-1}^k < n \leq W_{V_k(n)}^k$$

and dividing by $V_k(n)$ we get

$$\frac{W_{V_k(n)-1}^k}{V_k(n)} < \frac{n}{V_k(n)} \leq \frac{W_{V_k(n)}^k}{V_k(n)} \tag{1}$$

Note that $P_i$-a.s., $V_k(n) \to \infty$ as $n \to \infty$, and

$$\frac{W_n^k}{n} \to m_k = \mathbf{E}\left[T_1^k\right] \quad \text{as } n \to \infty \text{ by the strong law of large numbers.}$$

Now eqn.(1) can rewritten as

$$\frac{V_k(n)-1}{V_k(n)-1}\frac{W^k_{V_k(n)-1}}{V_k(n)} \;<\; \frac{n}{V_k(n)} \;\leq\; \frac{W^k_{V_k(n)}}{V_k(n)}$$

or equivalently

$$\frac{V_k(n)-1}{V_k(n)}\frac{W^k_{V_k(n)-1}}{V_k(n)-1} \;<\; \frac{n}{V_k(n)} \;\leq\; \frac{W^k_{V_k(n)}}{V_k(n)}$$

Combining the above we conclude that the left side and the right side of the above inequality tend to $1/m_k$, $P_i$-a.s. as $n \to \infty$. Hence,

$$\frac{n}{V_k(n)} \to m_k \qquad P_i \text{ almost surely as } n \to \infty$$

$$\Rightarrow \frac{V_k(n)}{n} \to \frac{1}{m_k} = \pi_k \qquad P_i{-}a.s. \text{ as } n \to \infty, \text{where } \pi \text{ is the stationary distribution.}$$

Now,

$$\frac{V_k(n)}{n} = \frac{1}{n}\sum_{l=0}^{n-1} \mathbf{1}_{\{X_l=k\}}$$

is the fraction of time that $X$ spends in state $k$ in the interval $[0, n-1]$, and so $1/m_k$ is the "long run fraction of time $X$ spends in $k$."

By the bounded convergence theorem we have that

$$\mathbf{E}_i\left[\frac{1}{n}\sum_{l=0}^{n-1} \mathbf{1}_{\{X_l=k\}}\right] \to \frac{1}{m_k} \qquad \text{as } n \to \infty$$

$$\text{i.e.,}\quad \frac{1}{n}\sum_{l=0}^{n-1} \mathbf{E}_i\left[\mathbf{1}_{\{X_l=k\}}\right] \to \frac{1}{m_k} \qquad \text{as } n \to \infty$$

$$\Longleftrightarrow \frac{1}{n}\sum_{l=0}^{n-1} P_i\left(X_l = k\right) \to \frac{1}{m_k} \qquad \text{as } n \to \infty$$

$$\Longleftrightarrow \frac{1}{n}\sum_{l=0}^{n-1} \left(\mathbf{P}^l\right)_{ik} \to \frac{1}{m_k} \qquad \text{as } n \to \infty, \tag{2}$$

i.e., the Cesaro averages of the sequence $\left\{\left(\mathbf{P}^l\right)_{ik}\right\}_{l=0}^{\infty}$ converge to $1/m_k$ as $n \to \infty$. *Note: eqn.(2) pertains whether the Markov chain is aperiodic or not.* Also by a coupling argument one can show that if the Markov chain is aperiodic then $\lim_{n\to\infty} \mathbf{P}^n_{ik}$ exists and does not depend on $i$. Since the above Cesaro averages converge to $1/m_k$ then $\lim_{n\to\infty} \mathbf{P}^n_{ik} = 1/m_k$.

## 2 Reducible Markov Chains

For a reducible Markov chain the transition matrix $\mathbf{P}$ can be arranged in partitioned diagonal form

$$\mathbf{P} \quad = \quad \left[\begin{array}{cccc|c} \mathbf{P}_1 & 0 & \cdots & 0 & \\ 0 & \mathbf{P}_2 & 0 & \vdots & \mathbf{\Theta} \\ \vdots & \ddots & \ddots & 0 & \\ 0 & \cdots & 0 & \mathbf{P}_k & \\ \hline & & \mathbf{S} & & \mathbf{Q} \end{array}\right]$$

where the block of diagonal $\mathbf{P}_i$ matrices, $1 \leq i \leq k$, is the transition matrix from recurrent to recurrent states (note that there could be infinitely many matrices in the block, i.e., $k$ could be $\infty$), $\mathbf{S}$ is the transition matrix from transient to recurrent states, $\mathbf{\Theta}$ is the zero matrix since recurrent states cannot enter transient states, and $\mathbf{Q}$ is the transition matrix of transient to transient states. Then

$$\mathbf{P}^n \quad = \quad \left[\begin{array}{cccc|c} \mathbf{P}_1^n & 0 & \cdots & 0 & \\ 0 & \mathbf{P}_2^n & 0 & \vdots & \mathbf{\Theta} \\ \vdots & \ddots & \ddots & 0 & \\ 0 & \cdots & 0 & \mathbf{P}_k^n & \\ \hline & & \mathbf{S}_n & & \mathbf{Q}^n \end{array}\right].$$

*Note:* $\mathbf{S}_n \neq \mathbf{S}^n$.

## 3 Reversible Markov Chains

If there is a probability distribution $\pi$ such that

$$\pi_i P_{ij} = \pi_j P_{ji} \quad \text{for all } i, j, \qquad \text{(called “detailed balance”)}$$

then the Markov chain is time reversible with transition matrix $\mathbf{P}$. Here an intuitive picture is to think of the transition probability, $\pi_i P_{ij}$ , as a “mass flow” from $i$ to $j$

$$\underset{i}{\odot} \xrightarrow{\pi_i P_{ij}} \underset{j}{\odot}$$

to be in balance with the “mass flow” from $j$ to $i$

$$\underset{i}{\odot} \xleftarrow{\pi_j P_{ji}} \underset{j}{\odot}$$

as $n \to \infty$ (i.e., mass conservation in stationarity).

The following properties hold if detailed balance is satisfied by a probability distribution $\pi$.

(i) $\pi$ is a stationary distribution, i.e., $\pi' = \pi'\mathbf{P}$.

Proof: Fix $i$, then by detailed balance above we have that

$$\pi_i P_{ij} = \pi_j P_{ji} \quad \forall j$$

then summing over all $j$ we get

$$\sum_j \pi_i P_{ij} = \sum_j \pi_j P_{ji} \Longrightarrow \pi_i \sum_j P_{ij} = \sum_j \pi_j P_{ji} \Longrightarrow \pi_i = \sum_j \pi_j P_{ji}$$

since $\sum_j P_{ij} = 1$. Noticing the above is just matrix-vector multiplication, hence $\pi' = \pi'\mathbf{P}$.

(ii) If the Markov chain is initialized with the stationary distribution $\pi$ then the Markov chain is time reversible, i.e., for any fixed time $N$, let

$$\tilde{X}_n = X_{N-n}, \qquad n = 0, 1, 2, \ldots, N,$$

then $\tilde{X}$ is Markov($\pi, \mathbf{P}$).

Proof: First prove $\tilde{X}$ is Markov. Fix $n < N$, and let $i_0, i_1, i_2, \ldots, i_n, i_{n+1} \in \mathcal{S}$, where $\mathcal{S}$ is the state space of the process. Then

$$\begin{aligned}
&P\left(\tilde{X}_{n+1} = i_{n+1} \mid \tilde{X}_n = i_n, \tilde{X}_{n-1} = i_{n-1}, \ldots, \tilde{X}_0 = i_0\right) \\
&= \frac{P\left(X_{N-n-1} = i_{n+1}, X_{N-n} = i_n, X_{N-n+1} = i_{n-1}, \ldots, X_N = i_0\right)}{P\left(X_{N-n} = i_n, X_{N-n+1} = i_{n-1}, \ldots, X_N = i_0\right)} \\
&= \frac{\pi_{i_{n+1}} P_{i_{n+1},i_n} P_{i_n,i_{n-1}} \cdots P_{i_1,i_0}}{\pi_{i_n} P_{i_n,i_{n-1}} P_{i_{n-1},i_{n-2}} \cdots P_{i_1,i_0}} \\
&= \frac{\pi_{i_{n+1}}}{\pi_{i_n}} P_{i_{n+1},i_n} \\
&= \frac{P\left(X_{N-n-1} = i_{n+1}, X_{N-n} = i_n\right)}{P\left(X_{N-n} = i_n\right)} \\
&= P\left(\tilde{X}_{n+1} = i_{n+1} \mid \tilde{X}_n = i_n\right) \\
&= \tilde{P}_{i_n,i_{n+1}} \\
\Longrightarrow\ & \tilde{X}_n \text{ is Markov.}
\end{aligned}$$

Now prove $\tilde{X}$ is Markov($\pi, \mathbf{P}$). From above we have

$$\tilde{P}_{i_n,i_{n+1}} = \frac{\pi_{i_{n+1}}}{\pi_{i_n}} P_{i_{n+1},i_n} = \frac{\pi_{i_n}}{\pi_{i_n}} P_{i_n,i_{n+1}}$$

where the last equality is from using detailed balance. Therefore,

$$\implies \tilde{P} = P.$$

Also, $\tilde{X}$ has initial distribution given by distribution of $X_N$, i.e., $\pi$.

**Theorem:** If a Markov chain is irreducible, positive recurrent, and its transition graph is a tree, then the Markov chain is reversible.

**Proof:** see Frank Kelly's book on Reversibility and Stochastic Networks.

**Example:** Let $X$ be the reflecting random walk, with reflecting state at $i = 0$, on the state space of non-negative integers, with transition probabilities

$$P_{i,i-1} = q, \qquad \text{for } i \geq 1$$

$$P_{i,i+1} = p, \qquad \text{for } i \geq 1$$

$$P_{0,0} = q, \qquad P_{0,1} = p \qquad \text{for } i = 0,$$

and $0 < p < q,\ p + q = 1$. $X$ has only one communicating class, and positive recurrent since $p < q$, and we can check to see that there is a probability distribution $\pi$ satisfying detailed balance:

$$\pi_i P_{i,i+1} = \pi_{i+1} P_{i+1,i} \qquad i = 0, 1, 2, 3, \ldots$$

$$\implies \pi_{i+1} = \frac{p}{q}\pi_i \qquad \implies \pi_i = \left(\frac{p}{q}\right)^i \pi_0$$

and with the constraint $\sum_i \pi_i = \sum_i \left(\frac{p}{q}\right)^i \pi_0 = 1 \quad \implies \pi_0 = 1 - p/q$.

# 4 Hidden Markov Models

Hidden Markov models have an observed output process $Y$, where $Y_n = f(X_n, \xi_n)$ is a function of a discrete time Markov chain $X$, and some additional random process $\xi$. It is best explained first through an example.

**Example: Unfair Casino**

Suppose a gambler is gambling against the house by a simple betting process involving rolling a die. Here the observable outputs are the outcomes of the die throws $Y_n \in \{1, 2, 3, 4, 5, 6\}$. But suppose the gambler does not know whether the die being used is fair or unfair, that is suppose a fair die has an *output* probability distribution with probability $p_i = 1/6$, for $i = 1, 2, 3, 4, 5, 6$, and an unfair die has *output* distribution: $p = (1/6, 1/6, 1/6, 1/6, 1/4, 1/12)$. Let the process that registers whether the house is using a fair or unfair die be $X$, i.e.

$$X_n = 1 \text{ if fair die}, \quad X_n = 2 \text{ if unfair die}$$

and suppose the transition matrix for $X$ is

$$\mathbf{P} = \begin{bmatrix} 5/6 & 1/6 \\ 2/3 & 1/3 \end{bmatrix}$$

In particular, this situation corresponds to where there is a probability of 5/6 that the house keeps using a fair die in the next round, and probability 1/6 of switching to an unfair die. *Using a fair or unfair die is registered by the hidden process,* $X$*, and the additional random process,* $\xi = (\xi_n)$*, has distribution at time* $n$ *given by* $p_i = 1/6$ *for all* $i$ *if the fair die is in use, and given by* $p = (1/6, 1/6, 1/6, 1/6, 1/4, 1/12)$ *if the unfair die is in use at time* $n$*. The observable output,* $Y$*, records the outcome of the roll of the die at each time step.*