MATH 285A- Lecture Notes #8

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30 April 2007

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1 Warning

Invariant Measure :

$$\begin{aligned} \lambda &= (\lambda_i)_{i\in S} \\ \lambda &\geq 0, \quad \lambda \neq 0 \\ \lambda' P &= \lambda'. \end{aligned}$$

Invariant Distribution: (Invariant probability distribution, stationary distribution)

$$\begin{aligned} \pi &= (\pi_i)_{i\in S} \\ \pi_i &\geq 0, \quad \forall i, \quad \sum_{i\in S} \pi_i = 1 \\ \pi' P &= \pi' \end{aligned}$$

2 Hw Problem-last question (hint)

Show that the Markov chain in Exercise 1.3.4 is transient and deduce that $P_i(X_n \to \infty \quad as \quad n \to \infty) = 1$ given $X_0 = i$ for any $i \geq 1$.

We want to prove that

$$\begin{aligned} P_i(X_n \to \infty \quad as \quad n \to \infty) &= 1, \quad \forall i \in S = \{0, 1, 2, ...\} \\ i.e \quad X_n \to \infty \quad as \quad n \to \infty \quad & P_i - a.s, \quad \forall i \qquad (1) \end{aligned}$$

Fix i and focus on (1).

i.e., we want to prove that $X_n(w) \to \infty \quad as \quad n \to \infty$ for almost every w under $P_i(\forall i)$.

+——+——+——+——+——+——+——+——+—— time line
0 1 2 .... k

Want to show that for each $k \geq 1$, $X_n(w) > k$ for all n sufficiently large for almost every w under P_i, i.e., X_n should visit states in [0,k] only finitely many times P_i-a.s.

Want to prove $P_i(X_n$ is eventually not in [0,k]) =1 for each $k \geq 1, (\forall i)$

Read Ch 1.4, 1.5-1.10.

3 Strong Markov Property

X Markov process (λ, P).

3.1 Stopping Times (Optional Times)

A stopping time (with respect to X) is a function $T : \Omega \to \{0, 1, 2...\} \cup \{\infty\}$ such that for each $m = 0, 1, 2...$

$\{T = m\} = \{w \in \Omega : T(w) = m\}$ depends only on $X_0, X_1, ..., X_m$.

Example:

Fix state j: $T^j = inf\{n \geq 1 : X_n = j\}$ is a stopping time.

Check
Fix $m \in \{0, 1, 2, ...\}$

$$\begin{aligned} \{T^j = m\} &= \emptyset, & m = 0. \\ & \{X_1 \neq j, X_2 \neq j, ..., X_{m-1} \neq j, X_m = j\} & m \geq 1. \end{aligned}$$

The following is not in general a stopping time: $T_i = sup\{n \geq 1 : X_n = i\}$ = last time visit i.
To determine the event $\{T_i = m\}$, you need to know that happens to X after m, and so this is not a stopping time in general.

3.2 Strong Markov Property

Let T be a stopping time relative to the Markov chain X. Conditioned on $T < \infty$ and $X_T = i$, $(X_{T+n})_{n=o}^{\infty}$ is a Markov chain with parameters (δ_i, P). δ_i = point mass at i, i.e.,

$$P(X_{T+n} = j_n,\ X_{T+n-1} = j_{n-1}, ..., X_T = j_0 \quad | \quad X_0 = i_0, X_1 = i_1, ..., X_T = i, T < \infty). \tag{2}$$

some event in the future beyond T event in the past up to $T < \infty$.

$$= P_i(X_n = j_n,\ X_{n-1} = j_{n-1}, ..., X_0 = j_0), \quad \forall\ n \geq 0, j_0, j_1, ...j_n, \quad i_0, i_1, ...i.$$

Proof:

$$\{T < \infty\} = \bigcup_{m=0}^{\infty} \{T = m\}$$

Want to prove (2). It is equivalent to :

$$\begin{aligned} &P(X_{T+n} = j_n,\ X_{T+n-1} = j_{n-1}, ..., X_T = j_0, X_0 = i_0, X_1 = i_1, ..., X_T = i, T < \infty) \\ &= P_i(X_n = j_n,\ X_{n-1} = j_{n-1}, ..., X_0 = j_0) * P(X_0 = i_0, X_1 = i_1, ..., X_T = i, T < \infty). \end{aligned}$$

Enough to show for each $m = 0, 1, 2, ...$

$$\begin{aligned} & P(X_{T+n} = j_n,\ X_{T+n-1} = j_{n-1}, ..., X_T = j_0; X_0 = i_0, X_1 = i_1, ..., X_T = i, T = m) \\ &= P_i(X_n = j_n,\ X_{n-1} = j_{n-1}, ..., X_0 = j_0) * P(X_0 = i_0, X_1 = i_1, ..., X_T = i, T = m). \end{aligned} \tag{3}$$

In time subscripts, can replace T by m in the above on $\{T = m\}$ and then (3) follows by regular Markov property. □

3.3 Stationary Distribution

Assume M.C. is irreducible and positive recurrent (not necessarily aperiodic). Let

$$T_k = inf\{n \geq 1 : X_n = k\}$$

T_k: time to return to k.

$$\gamma_i^k = E_k\left[\sum_{n=0}^{T_k-1} 1_{\{X_n=i\}}\right], \quad k \in \mathbb{S}, i \in \mathbb{S}$$

γ_i^k: amount of time you spend in i before come back to k.
By Theorem 1.7.5 we have the following: $\gamma_k^k = 1$ and for

$$\gamma^k = \begin{pmatrix} \gamma_0^k \\ \gamma_1^k \\ . \\ . \\ . \end{pmatrix},$$

$$\gamma^k P = \gamma^k, \quad (invariant).$$

Since X is positive recurrent,

$$\begin{aligned} \sum_{i \in \mathbb{S}} \gamma_i^k &= \sum_{i \in \mathbb{S}} E_k \left[\sum_{n=0}^{T_k-1} 1_{\{X_n=i\}}\right] \\ &= E_k \left[\sum_{i \in \mathbb{S}} \sum_{n=0}^{T_k-1} 1_{\{X_n=i\}}\right] \qquad (4) \\ &= E_k \left[\sum_{n=0}^{T_k-1} \sum_{i \in \mathbb{S}} 1_{\{X_n=i\}}\right] \\ &= E_k \left[\sum_{n=0}^{T_k-1} 1\right] \qquad (5) \\ &= E_k\left[T_k\right] = m_k < \infty \quad (because\ positive\ recurrent) \end{aligned}$$

So,

$$\pi_i = \frac{\gamma_i^k}{m_k}, \quad i \in \mathbb{S}.$$

is a stationary distribution. (normalized for elements to sum to 1.)
Irreducibility gives that π is unique (needs an argument - see text). Then we must have $\pi_k = \frac{1}{m_k}$ for each $k \in \mathbb{S}$, is the unique stationary distribution.

3.4 ERGODIC Thm:

Assume M.C. is positive recurrent and irreducible.

Fix starting state i.

Fix state $k \in \mathbb{S}$,

T_0^k T_1^k T_2^k ....

$\curvearrowright$ $\curvearrowright$

+———+———+———+———+———+———+———+——— time line

W_{-1}^k W_0^k W_1^k ....

$$T_0^k = inf\{n \geq 0 : X_n = k\}$$

first time, X is in k.

T_1^k : amount of time between the first and 2nd visits to k. T_2^k : amount of time between the 2nd and 3rd visits to k ..... Let

$$W_n^k = \sum_{l=0}^{n} T_l^k$$

waiting time till the $(n+1)$th visit to k. Let

$$W_{-1}^k = 0.$$

By the strong Markov property $\{ T_l^k \}_{l=1}^{\infty}$ are $i.i.d$ with finite mean $m_k = E_i [T_1^k]$.

S.L.L.N (Strong Law of Large Nos)

$P_i - a.s.$

$$\frac{1}{n}\sum_{l=1}^{n} T_l^k \rightarrow m_k \quad as \quad n \rightarrow \infty.$$

So,

$$\frac{W_n^k}{n} \rightarrow m_k \quad as \quad n \rightarrow \infty, \quad P_i - a.s$$

What was the time of the last visit to k before n?

———+——————+——————+——————+——— time line

$W_{V_k(n)-1}^k$ $n\ fixed$ $W_{V_k(n)}^k$

$$V_k(n) = \sum_{l=0}^{n-1} 1_{\{X_l = k\}} \quad \rightarrow \infty \quad as\ n \rightarrow \infty \quad (by\ recurrence)$$

$V_k(n)$: number of visits to k up to time n-1.

$$W^k_{V_k(n)-1} \leq n-1 < W^k_{V_k(n)}$$

so

$$W^k_{V_k(n)-1} < n \leq W^k_{V_k(n)}$$

Divide by $V_k(n)$,

$$\frac{W^k_{V_k(n)-1}}{V_k(n)} < \frac{n}{V_k(n)} \leq \frac{W^k_{V_k(n)}}{V_k(n)}$$

Both the extreme left and extreme right expressions tend to m_k P_i-a.s., as $n \to \infty$, because $V_k(n)$ tends to infinity P_i-a.s. as $n \to \infty$, and W^k_ℓ/ℓ tends to m_k P_i-a.s. as ℓ tends to infinity, hence the composition tends to m_k, P_i-a.s. as $n \to \infty$.

Thus,

$$\frac{n}{V_k(n)} \to m_k \quad as \quad n \to \infty \quad P_i - a.s.$$

$$\frac{V_k(n)}{n} \to \frac{1}{m_k} \quad as \quad n \to \infty \quad P_i - a.s.$$