

Read Ch 1.4, 1.5-1.10.

3 Strong Markov Property

X Markov process (λ, P) .

3.1 Stopping Times (Optional Times)

A stopping time (with respect to X) is a function $T : \Omega \rightarrow \{0, 1, 2, \dots\} \cup \{\infty\}$ such that for each $m = 0, 1, 2, \dots$

$\{T = m\} = \{\omega \in \Omega : T(\omega) = m\}$ depends only on X_0, X_1, \dots, X_m .

Example:

Fix state j : $T^j = \inf\{n \geq 1 : X_n = j\}$ is a stopping time.

Check

Fix $m \in \{0, 1, 2, \dots\}$

$$\begin{aligned} \{T^j = m\} &= \emptyset, & m = 0. \\ & \{X_1 \neq j, X_2 \neq j, \dots, X_{m-1} \neq j, X_m = j\} & m \geq 1. \end{aligned}$$

The following is not in general a stopping time: $T_i = \sup\{n \geq 1 : X_n = i\} =$ last time visit i .
To determine the event $\{T_i = m\}$, you need to know that happens to X after m , and so this is not a stopping time in general.

3.2 Strong Markov Property

Let T be a stopping time relative to the Markov chain X . Conditioned on $T < \infty$ and $X_T = i$, $(X_{T+n})_{n=0}^\infty$ is a Markov chain with parameters (δ_i, P) . δ_i = point mass at i , i.e.,

$$P(X_{T+n} = j_n, X_{T+n-1} = j_{n-1}, \dots, X_T = j_0 \mid X_0 = i_0, X_1 = i_1, \dots, X_T = i, T < \infty). \quad (2)$$

some event in the future beyond T

event in the past up to $T < \infty$.

$$= P_i(X_n = j_n, X_{n-1} = j_{n-1}, \dots, X_0 = j_0), \quad \forall n \geq 0, j_0, j_1, \dots, j_n, \quad i_0, i_1, \dots, i.$$

Proof:

$$\{T < \infty\} = \bigcup_{m=0}^{\infty} \{T = m\}$$

Want to prove (2). It is equivalent to :

$$\begin{aligned} & P(X_{T+n} = j_n, X_{T+n-1} = j_{n-1}, \dots, X_T = j_0, X_0 = i_0, X_1 = i_1, \dots, X_T = i, T < \infty) \\ &= P_i(X_n = j_n, X_{n-1} = j_{n-1}, \dots, X_0 = j_0) * P(X_0 = i_0, X_1 = i_1, \dots, X_T = i, T < \infty). \end{aligned}$$

Enough to show for each $m = 0, 1, 2, \dots$

$$\begin{aligned} & P(X_{T+n} = j_n, X_{T+n-1} = j_{n-1}, \dots, X_T = j_0; X_0 = i_0, X_1 = i_1, \dots, X_T = i, T = m) \\ &= P_i(X_n = j_n, X_{n-1} = j_{n-1}, \dots, X_0 = j_0) * P(X_0 = i_0, X_1 = i_1, \dots, X_T = i, T = m). \end{aligned} \quad (3)$$

In time subscripts, can replace T by m in the above on $\{T = m\}$ and then (3) follows by regular Markov property. \square

3.3 Stationary Distribution

Assume M.C. is irreducible and positive recurrent (not necessarily aperiodic). Let

$$T_k = \inf\{n \geq 1 : X_n = k\}$$

T_k : time to return to k .

$$\gamma_i^k = E_k \left[\sum_{n=0}^{T_k-1} 1_{\{X_n=i\}} \right], \quad k \in \mathbb{S}, i \in \mathbb{S}$$

γ_i^k : amount of time you spend in i before come back to k .

By Theorem 1.7.5 we have the following: $\gamma_k^k = 1$ and for

$$\gamma^k = \begin{pmatrix} \gamma_0^k \\ \gamma_1^k \\ \vdots \\ \vdots \end{pmatrix},$$

$$\gamma^k P = \gamma^k, \quad (\text{invariant}).$$

Since X is positive recurrent,

$$\begin{aligned} \sum_{i \in \mathbb{S}} \gamma_i^k &= \sum_{i \in \mathbb{S}} E_k \left[\sum_{n=0}^{T_k-1} 1_{\{X_n=i\}} \right] \\ &= E_k \left[\sum_{i \in \mathbb{S}} \sum_{n=0}^{T_k-1} 1_{\{X_n=i\}} \right] \end{aligned} \quad (4)$$

$$\begin{aligned} &= E_k \left[\sum_{n=0}^{T_k-1} \sum_{i \in \mathbb{S}} 1_{\{X_n=i\}} \right] \\ &= E_k \left[\sum_{n=0}^{T_k-1} 1 \right] \end{aligned} \quad (5)$$

$$= E_k [T_k] = m_k < \infty \quad (\text{because positive recurrent})$$

So,

$$\pi_i = \frac{\gamma_i^k}{m_k}, \quad i \in \mathbb{S}.$$

is a stationary distribution. (normalized for elements to sum to 1.)

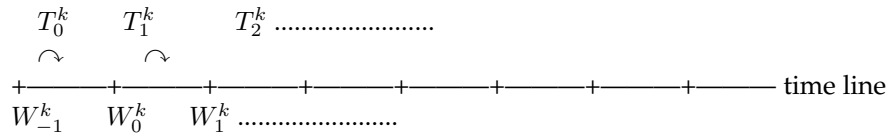
Irreducibility gives that π is unique (needs an argument - see text). Then we must have $\pi_k = \frac{1}{m_k}$ for each $k \in \mathbb{S}$, is the unique stationary distribution.

3.4 ERGODIC Thm:

Assume M.C. is positive recurrent and irreducible.

Fix starting state i .

Fix state $k \in \mathbb{S}$,



$$T_0^k = \inf\{n \geq 0 : X_n = k\}$$

first time, X is in k .

T_1^k : amount of time between the first and 2nd visits to k . T_2^k : amount of time between the 2nd and 3rd visits to k Let

$$W_n^k = \sum_{l=0}^n T_l^k$$

waiting time till the $(n + 1)$ th visit to k . Let

$$W_{-1}^k = 0.$$

By the strong Markov property $\{T_l^k\}_{l=1}^\infty$ are *i.i.d* with finite mean $m_k = E_i [T_1^k]$.

S.L.L.N (Strong Law of Large Nos)

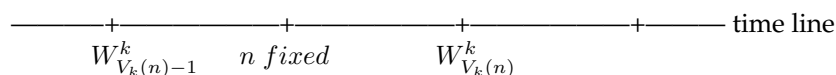
$P_i - a.s.$

$$\frac{1}{n} \sum_{l=1}^n T_l^k \rightarrow m_k \text{ as } n \rightarrow \infty.$$

So,

$$\frac{W_n^k}{n} \rightarrow m_k \text{ as } n \rightarrow \infty, \quad P_i - a.s$$

What was the time of the last visit to k before n ?



$$V_k(n) = \sum_{l=0}^{n-1} 1_{\{X_l=k\}} \rightarrow \infty \text{ as } n \rightarrow \infty \text{ (by recurrence)}$$

$V_k(n)$: number of visits to k up to time $n-1$.

$$W_{V_k(n)-1}^k \leq n - 1 < W_{V_k(n)}^k$$

so

$$W_{V_k(n)-1}^k < n \leq W_{V_k(n)}^k$$

Divide by $V_k(n)$,

$$\frac{W_{V_k(n)-1}^k}{V_k(n)} < \frac{n}{V_k(n)} \leq \frac{W_{V_k(n)}^k}{V_k(n)}$$

Both the extreme left and extreme right expressions tend to m_k P_i -a.s., as $n \rightarrow \infty$, because $V_k(n)$ tends to infinity P_i -a.s. as $n \rightarrow \infty$, and W_ℓ^k/ℓ tends to m_k P_i -a.s. as ℓ tends to infinity, hence the composition tends to m_k , P_i -a.s. as $n \rightarrow \infty$.

Thus,

$$\frac{n}{V_k(n)} \rightarrow m_k \quad \text{as } n \rightarrow \infty \quad P_i - a.s.$$

$$\frac{V_k(n)}{n} \rightarrow \frac{1}{m_k} \quad \text{as } n \rightarrow \infty \quad P_i - a.s.$$