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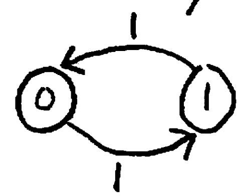
LECTURE 7

Math 285 4/23/07

Last time we showed a steady state distribution is a stationary distribution.

The converse is not always true. Examples:

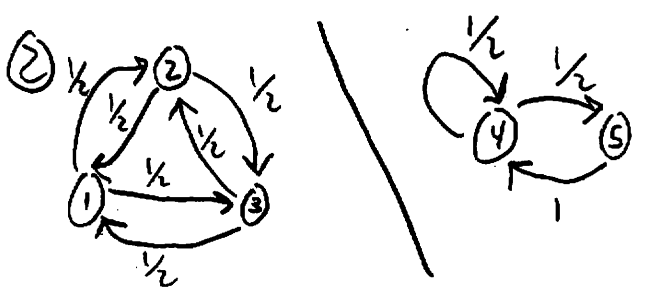
① Periodicity



$\pi = \left(\frac{1}{2}, \frac{1}{2}\right)$

$\pi' P = \pi'$, so we have a stationary distribution, but $\lim_{n\to\infty} P^n_{00}$ does not exist since the values oscillate between 0 and 1. Therefore π is not a steady state distribution.

②



$\pi = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0\right)$ and $\pi = \left(0, 0, 0, \frac{2}{3}, \frac{1}{3}\right)$ are stationary, and $\lim_{n\to\infty} P^n_{ij}$ exists, but it depends on i, in general, since it depends on which communicating class we start in. Therefore neither π is ~~not~~ a steady state distribution.

Defn: A Markov chain is irreducible if all its states belong to a single communicating class (this will necessarily be closed)

Sufficient Condition for irreducibility:

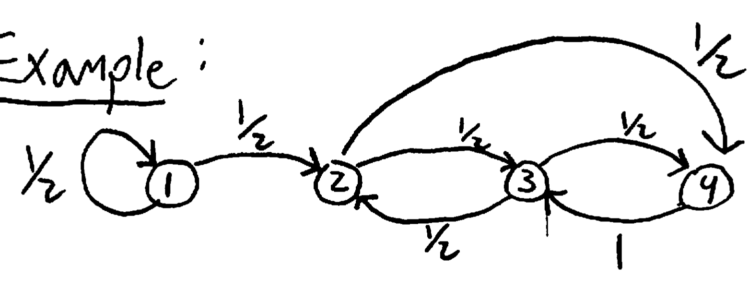
$P^n > 0$ (all entries of P are strictly positive)

Defn: The period of a state i is the greatest common divisor of all $n \geq 1$ s.t. $P^n_{ii} > 0$ (If $P^n_{ii} = 0$ for all $n \geq 1$ the period of i is defined to be 0).

$d(i) :=$ period of i

Fact: All states in the same communicating class have the same period

Example:



Communicating classes: $\{1\}$, $\{2,3,4\}$

$d(1)=1$ since $p_{11}^{1}>0$

$d(2)=d(3)=d(4)$:

$$p_{22}^{2}=\tfrac{1}{2}\cdot\tfrac{1}{2}>0$$

$$p_{22}^{3}=\tfrac{1}{2}\cdot 1\cdot\tfrac{1}{2}>0$$

$\gcd(2,3)=1$, so $d(2)=1=d(3)=d(4)$

If $d(i)>1$, then $p_{ii}^{n}=0$ for any n which is not a multiple of $d(i)$.

Transient & recurrent states

A state i is recurrent if
$P_i(X_n=i \text{ for some } n\geq 1)=1$

(In fact, i is recurrent iff
$P_i(X_n=i \text{ for infinitely many } n)=1$)

A state i is transient if
$P_i(X_n=i \text{ for some } n\geq 1)<1$

(i.e. $P_i(X_n\neq i \text{ for } \cancel{\text{any}}\ \text{all } n\geq 1)>0$)

(In fact, i is transient iff
$P_i(X_n = i \text{ for infinitely many } n) = 0$)

Fact: All states in a communicating class have the same recurrence/transience classification)

Revisiting the example above, we see that

$\{1\}$ is transient, since $P_1(X_n \neq 1 \text{ for all } n \geq 1) = P^1_{12} = \frac{1}{2} > 0$

$\{2,3,4\}$ is reccurrent, from the fact below:

Fact: A finite state Markov chain always has at least one recurrent communicating class

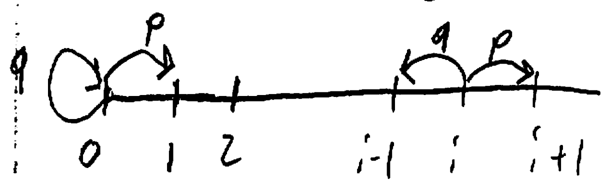
i is recurrent if
$P_i(T_i < \infty) = 1$
where $T_i = \inf\{n \geq 1 : X_n = i\}$ (first passage time to i)

If we have finitely many states and i is recurrent then $E_i[T_i] < \infty$.

If we have infinitely many states,
i is positive recurrent if $E_i[T_i] < \infty$
i is null recurrent if it is recurrent but $E_i[T_i] = \infty$

Fact: All states in a communicating class are either (i) transient, (ii) null recurrent, or (iii) positive recurrent

Example: reflecting random walk



$p, q > 0$, $p + q = 1$

$p > q$: Every state is transient, since $P_0(T_0 < \infty) < 1$ from our previous analysis

$p = q$: All states are recurrent since $P_0(T_0 < \infty) = 1$
We can use the fact that $E_i[H^{\{0\}}] = \infty$ $(i \neq 0)$ to see that our states are in fact null recurrent.

$p < q$: All states are recurrent since $P_0(T_0 < \infty) = 1$, and in fact are positive recurrent since $E_0(T_0) < \infty$

Basic Limit Theorem for Markov Chains

Suppose $X = (X_n, n \geq 0)$ is an irreducible, aperiodic ($d(i) = 1$ for all i) positive recurrent Markov chain.

Then $\lim_{n \to \infty} P_{ij}^n$ exists for all $i, j \in S$ and

$$\lim_{n\to\infty} P_{ij}^{n} = \pi_j \quad \forall i, j$$

where $\pi = (\pi_i)_{i\in S}$ is the unique stationary distribution for the Markov Chain.

In fact, $\pi_j = \dfrac{1}{E_j[T_j]} \quad \forall j \in S,$

More generally, suppose X is irreducible (but not necessarily aperiodic or pos. recurrent)

Recurrence classification	Stationary Distribution	Steady State Dist
Null Recurrent or transient	No	$\lim_{n\to\infty} P_{ij}^{n} = 0$
Pos. Recurrent	Yes (π)	aperiodic: $\lim_{n\to\infty} P_{ij}^{n} = \pi_j$ periodic (d): $\lim_{n\to\infty} P_{ii}^{nd} = d\pi_i$