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LECTURE 6

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For $q \neq p$:

General sol'n of $BK = \mathbb{1}$ is

$$k_i = a\left(\frac{q}{p}\right)^i + b + \frac{i}{q-p}$$

Use $k_0 = 0 = k_N$ to uniquely determine a and b.

$$k_i = \frac{N \frac{1}{q-p}}{\left(1-(q/p)^N\right)}\left(\left(\frac{q}{p}\right)^i - 1\right) + \frac{i}{q-p}$$

For $q = p$

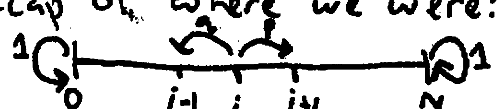
General sol'n of $Bk = \mathbb{1}$ is

$$k_i = ai + b\mathbb{1} - i^2 \qquad i = 0,1,2,\ldots,N-1,N$$

Use $k_0 = 0 = k_N$ to find a and b

$$k_i = Ni - i^2$$

Recap of where we were: 4/18



Trying to compute $k_i^{\{0,N\}} = E_i\left[H^{\{0,N\}}\right]$, $i = 0,\ldots,N$
is first hitting time on this set.

These k_i are the minimal non-negative solution of:

$$(1)\begin{cases} k_i = 1 + p\,k_{i+1} + q\,k_{i-1}, & i = 1,\ldots,N-1 \\ k_0 = k_N = 0 & \text{(start in absorbing state, absorbed right away)} \end{cases}$$

The problem with the above result is that
$k_i = \infty$ is a solution, $i = 1,\ldots,N-1$

We proceed by trying to find $k = \begin{pmatrix} k_0 \\ \vdots \\ k_N \end{pmatrix}$ where the entries are finite, and (1) is satisfied.

If $\exists!$ k, then k is the minimal nonnegative solution ($\exists! :=$ there exists a unique).

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Write equation system (1) in matrix form:

Let B be the $(N-1)\times(N+1)$ matrix

$$
\begin{array}{c}
\begin{array}{ccccccccc} 0 & 1 & 2 & 3 & 4 & \dots & N-1 & N & N+1 \end{array} \\
B = \left[\begin{array}{ccccccccc}
-q & 1 & -p & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & -q & 1 & -p & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & -q & 1 & -p & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots & -q & 1 & -p
\end{array}\right]
\begin{array}{c} 1 \\ 2 \\ 3 \\ \vdots \\ N-1 \end{array}
\end{array}
$$

Then $\left.Bk = 1 = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}\right\}$ (2)

The equation $Bk = 1$ is "in-homogeneous" b/c the right-hand side is not set equal to zero.

Suppose $k^{(3)}$ is a particular solution of (2). If k is some solution of (2), then $k - k^{(3)}$ is a solution of $B(k - k^{(3)}) = 0$.

Since the rows of B are linearly independent, B has rank $N-1$. We have $N+1$ unknowns; so there are two degrees of freedom.

We can write k as

$$k = k^{(3)} + a k^{(1)} + b k^{(2)}$$

where $k^{(1)}$ and $k^{(2)}$ are two linearly independent solutions of the homogeneous equation $Bk = 0$.

We then use the facts that $\cancel{k_0 = k_N}$ $k_0 = 0$ and $k_N = 0$ to find a and b.

Note that nothing done so far depends on q and p not depending on i.

This concludes our recap of where we were.

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We find:

If $p \neq q$, $k_i^{(1)} = \left(\frac{q}{p}\right)^i$, $k_i^{(2)} = 1$, $i = 0, 1, \ldots, N$

$k_i^{(3)} = \frac{i}{q-p}$, $i = 0, 1, \ldots, N$

If $p = q$ $(= 1/2)$, $k_i^{(1)} = i$, $k_i^{(2)} = 1$, $i = 0, 1, \ldots, N$

$k_i^{(3)} = -i^2$, $i = 0, 1, \ldots, N$

If $p \neq q$,
$$k_i = \frac{N}{1 - (q/p)^N} \cdot \frac{1}{q-p} \cdot \left(\left(\frac{q}{p}\right)^i - 1\right) + \frac{i}{q-p}$$
(note change from prior formula)

If $p = q$,
$$k_i = Ni - i^2 \quad \text{for } i = 0, 1, \ldots, N$$

Now,
$$k_i^{\{0,N\}} = E_i[H^{\{0,N\}}] \qquad i = 0, 1, \ldots, N$$
$$= E_i[H^{\{0\}} \wedge H^{\{N\}}] \quad \text{where } \wedge := \text{minimum operator.}$$

Note that $H^{\{N\}} \geq N - i$ P_i-as which goes to ∞ as $N \to \infty$.

Then
$$k_i^{\{0\}} = E_i[H^{\{0\}}]$$
$$= E_i[\lim_{N\to\infty} H^{\{0\}} \wedge H^{\{N\}}]$$
$$= \lim_{N\to\infty} E_i[H^{\{0\}} \wedge H^{\{N\}}] \quad \text{(by Monotone Converg. Thm.)}$$
$$= \lim_{N\to\infty} k_i^{\{0,N\}}$$

So now we ~~have~~ can investigate the k_i as $N \to \infty$.
There are three cases: $p = q$, $p > q$, $p < q$.

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Case 1: $p>q$

$k_i^{\{0\}} = \lim_{N\to\infty} k_i^{\{0,N\}} = +\infty$. (Refer to formula for $k_i^{\{0,N\}}$ and let $N\to\infty$)

for $i \geq 1$.

Case 2: $p=q$

$k_i^{\{0\}} = \lim_{N\to\infty} k_i^{\{0,N\}} = +\infty$ as long as $i>0$.

Case 3: $p<q$

$k_i^{\{0\}} = \lim_{N\to\infty} k_i^{\{0,N\}} = \dfrac{i}{q-p}$ for $i \geq 1$

(This is b/c the denominator in our expression for $k_i^{\{0,N\}}$ goes to $-\infty$ faster than the numerator goes to ∞)

Thus, the mean-time of hitting the origin is finite in games where $q>p$ (as in casinos).

Long-run Behavior of Markov Chains

– see text sections 1.5 – 1.8 (you can temporarily skip 1.4)

Let $X = \{X_n, n=0,1,\dots\}$ be a Markov chain with state space $\mathbb{S} \subseteq \{0,1,2,\dots\}$ ($\mathbb{S}$ will be a proper subset of $\mathbb{N}$ if the Markov chain is finite.

Steady-state (or limiting) probability distribution: a probability vector $\pi = (\pi_0, \pi_1, \dots)$ ~~with~~ (non-negative entries that sum to unity) such that $\lim_{n\to\infty} P_{ij}^n = \pi_j$ holds for all starting states $i \in \mathbb{S}$ and destination states $j \in \mathbb{S}$.

$P_{ij}^n := (P^n)_{ij} = P(X_n = j \mid X_0 = i)$

(5)

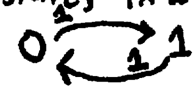
(5)

π_j is the long-run probability that the Markov chain is in state j.

~~This probability~~

A steady-state limiting probability may not always exist.

Examples:

(1) Simple model with two states in which the MC flips back and forth: 

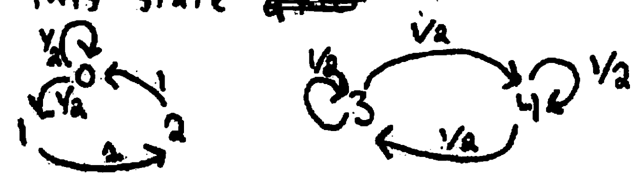
$$P_{11}^{n} = P(X_n = 1 \mid X_0 = 1) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

This sequence of probabilities does not converge.

~~Periodic~~

Periodic: Starting from a given state, you can only return to this state ~~after~~ at multiples of 2.

(2)



In this case, we have two closed communicating classes. The ~~f~~ limiting probabilities we seek will depend on where we start.

If $\lim_{n\to\infty} P_{11}^{n}$ exists, it will be different from $\lim_{n\to\infty} P_{31}^{n}$, which is zero.

Reducible: the above example is reducible in that we can consider the components of the state-space separately and not lose any of the meaningful analysis. More formally, reducible means more than one closed communicating class.

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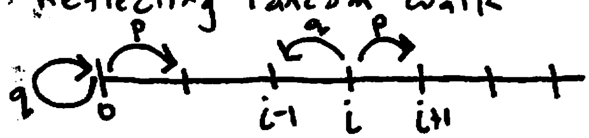
3. Reflecting random walk



, $p>0$, $q>0$, $p+q=1$

If $p>q$, $P_{11}^n \to 0$ as $n \to \infty$.
This is an example of a transient MC.

If $p=q$, $P_{11}^n \to 0$ as $n \to \infty$.
We know in this case that the random walker has a mean-time until returning to zero that is infinite even though he does ~~get~~ (eventually) ~~also~~ return to zero. In this case, he wanders around so long that the ~~probability~~ mean time of ~~being~~ returning to 1 is infinite ~~is zero~~.
This is an example of a null/recurrent MC.

~~The~~ A basic problem (mathematically) is:

$$0 = \sum_{j \in S} \left(\lim_{n \to \infty} P_{ij}^n \right) \neq \lim_{n \to \infty} \sum_{j \in S} P_{ij}^n = 1$$

when $|S| = \infty$ (when we have infinitely many states).
(Lebesgue's Dominated Convergence thm. does not apply)

Stationary Distribution

Probability distribution on S

$$\pi' P = \pi$$

$$\sum_{i \in S} P(X_1 = j \mid X_0 = i)\, \pi_i = \pi_j$$

In words, if π is the initial distribution of X, then π is also the distribution at each time $n = 1, 2, \ldots$

Contrast this with the notion of steady-state distribution: a stationary distribution starts as an initial distribution and always maintains this distribution.

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Theorem: If π is a steady-state distribution, then π is a stationary distribution. (But, not vice-versa)

Proof:

Suppose π is a steady-state distribution.

Fix $j \in S$.

$\pi_j = \lim_{n\to\infty} P^n_{kj} \quad \forall k \in S$. Also, $\pi_j = \lim_{n\to\infty} P^{n+1}_{kj} \quad \forall k \in S$.

$$= \lim_{n\to\infty} P^{n+1}_{kj}$$

$$= \lim_{n\to\infty}\Big(\sum_{i\in S} P^n_{ki} P_{ij}\Big)$$

$$\geq \lim_{n\to\infty} \sum_{\substack{i\in S\\ i\leq M}} P^n_{ki} P_{ij}$$

we shrank the state-space over which we're looking at transition probabilities.

(as see below, the limit actually exists)

$$= \sum_{\substack{i\in S\\ i\leq M}}\Big(\lim_{n\to\infty} P^n_{ki} P_{ij}\Big)$$

$$= \sum_{\substack{i\in S\\ i\leq M}} \pi_i P_{ij} \qquad \text{for any } M \geq 0$$

This inequality holds for all $M \geq 0$. So if we let $M \to \infty$, the inequality will still hold.

Thus,

$$\pi_j \geq \sum_{i\in S} \pi_i P_{ij} = (\pi P)_j$$

Actually have equality.

To see this, suppose $j_0 \in S$ with $\pi_{j_0} > (\pi P)_{j_0}$

Summing over $j \in S$ we have

$$1 = \sum_{j\in S} \pi_j > \sum_j (\pi P)_j = \sum_{j\in S}\sum_{i\in S} \pi_i P_{ij}$$

$$= \sum_{i\in S}\sum_{j\in S} \pi_i P_{ij} = \sum_i \pi_i \sum_j P_{ij} = \sum_i \pi_i = 1$$

This is a contradiction, so a s-s distribution is stationary.