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LECTURE 5

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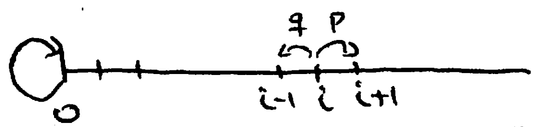
– Hitting Times AND Absorption Probabilities

$h_i^A = P_i(H^A < \infty)$, A = absorbing set $\subset S$, $i \in S$

$k_i^A = E_i[H^A]$, $i \in S$

Example: random walk on $\mathbb{N}$ with absorbing $\{0\}$



$S = \mathbb{N} = \{0, 1, \dots\}$

$p, q > 0$, $p + q = 1$

Compute h_i^A where $A = \{0\}$, $i \in S$.

$$\begin{cases} h_0^A = 1 \\ h_i^A = p h_{i+1}^A + q h_{i-1}^A & i \neq 0 \end{cases}$$

$u_i = h_{i-1} - h_i$, $i = 1, 2, \dots$

$p u_{i+1} = q u_i \quad \forall i = 1, 2, \dots$

$\Rightarrow u_2 = \frac{q}{p} u_1$

$\vdots$

$u_{i+1} = \left(\frac{q}{p}\right)^i u_1$

$u_1 = h_0^A - h_1^A = 1 - h_1^A$, so $u_{i+1} = \left(\frac{q}{p}\right)^i (1 - h_1^A)$, $i = 1, 2, \dots$

For $i = 1, 2, \dots$

$$h_i^A = (h_i^A - h_{i-1}^A) + (h_{i-1}^A - h_{i-2}^A) + \dots + (h_1^A - h_0^A) + h_0^A =$$

$$= (-u_i) + (-u_{i-1}) + \dots + (-u_1) + 1 =$$

$$= \left(-\left(\frac{q}{p}\right)^{i-1} - \left(\frac{q}{p}\right)^{i-2} \dots - \left(\frac{q}{p}\right)^0\right) u_1 + 1 =$$

$\Longrightarrow$

$$= -\sum_{j=0}^{i-1}\left(\frac{q}{p}\right)^j u_1 + 1$$

Note $h_i^A \in [0,1]$,

if $\frac{q}{p} \geq 1$, the only way for h_i^A to be $\in [0,1]$ $\forall i$ is for $u_1 = 0$

$$\Rightarrow h_i^A = 1 \quad \forall i.$$

if $\frac{q}{p} < 1$, then $\sum_{j=0}^{i-1}\left(\frac{q}{p}\right)^j \longrightarrow \frac{1}{1-\frac{q}{p}}$ as $i \to \infty$.

Need $h_i^A \geq 0$, so $1 - \left[\sum_{j=0}^{i-1}\left(\frac{p}{q}\right)^j\right] u_1 \geq 0 \quad \forall i$

$$\Rightarrow u_1 \sum_{j=0}^{i-1}\left(\frac{q}{p}\right)^j \leq 1 \Rightarrow u_1 \leq \frac{1}{\sum_{j=0}^{i-1}\left(\frac{q}{p}\right)^j} \quad \forall i$$

$$\Rightarrow u_1 \leq 1 - \frac{q}{p}$$

So $h_i^A = 1 - \left(\frac{1-\left(\frac{q}{p}\right)^i}{1-\frac{q}{p}}\right) \cdot u_1 \geq 1 - \frac{1-\left(\frac{q}{p}\right)^i}{1-\frac{q}{p}}\left(1-\frac{q}{p}\right)$

$$= \left(\frac{q}{p}\right)^i, \quad i = 0,1,2,\ldots$$

the minimal non-negative solution.

$A = \{0\}$

$k_i^A,\ i \in S$ are the min. non-neg. sol$^{\underline{n}}$s of

$$\begin{cases} k_0^A = 0 \\ k_i^A = 1 + p k_{i+1}^A + q k_{i-1}^A, & i = 1, 2, \ldots \end{cases}$$

(k_i^A may be infinite)

One way to proceed is to seek $k_i^A < \infty\ \forall i$.
(if $\nexists$ such sol$^{\underline{n}}$, then $k_i^A = +\infty\ \forall i \geq 1$ is a minimal sol$^{\underline{n}}$). if there is a unique finite soln then it is the minimal soln).

– If $p > q$, $h_i^A < 1$ for $i = 1, 2, \ldots$

$\Rightarrow$

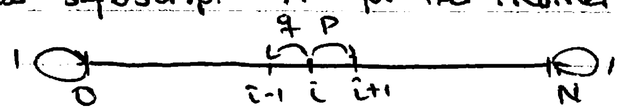
$P_i(H_i^A = \infty) > 0$ for $i = 1, 2, \ldots$

$\Rightarrow E_i(H^A) = \infty$ for $i = 1, 2, \ldots$

but can have $P_i(H_A < \infty) = 1\ \forall i \geq 1$

& $E_i(H_A] = \infty\ \forall i \geq 1$

Consider truncated problem
where $A = \{0, N\}$ & N is large, $i \in \{0, 1, 2, \ldots, N\}$
Suppress superscript A for the moment.



$k_0 = 0$, $k_N = 0$

& For $i = 1, 2, \ldots, N-1$,

$$k_i = 1 + p k_{i+1} + q k_{i-1} \qquad (1)$$

$\Rightarrow$

Unknowns are, $k_0, k_1, \dots, k_N$

where we put

$$k = \begin{pmatrix} k_0 \\ \vdots \\ k_N \end{pmatrix} \quad \& \quad B = \begin{array}{c} \begin{array}{cccccc} 0 & 1 & \cdots & & & N \end{array} \\ \begin{pmatrix} -q & 1 & -p & 0 & \cdots & 0 \\ 0 & -q & 1 & -p & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \\ 0 & \cdots & 0 & -q & 1 & -p \end{pmatrix} \end{array} \Bigg\} \; N-1 \text{ rows}$$

$(N-1)\times(N+1)$ matrix

$$\& \quad Bk = 1 = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \quad (N-1)\times 1$$

(corresponds to eqns in (1))

General solⁿ of $Bk=1$ looks like

$$k = a k^{(1)} + b k^{(2)} + k^{(3)}$$

where

$$B k^{(j)} = 0 \quad j=1,2 \qquad k^{(1)}, k^{(2)} \text{ are lin. indept.}$$

$$B k^{(3)} = 1 \qquad , \; a, b \in \mathbb{R}$$

Seek $k_i = \gamma^i$ $i=0,\dots,N$ for homog. eqtⁿ:

$$k_i = p k_{i+1} + q k_{i-1}, \quad i = 1, 2, \dots, N-1$$

Substitute $k_i = \gamma^i$

$$\Longrightarrow$$

$$\gamma^i = p\gamma^{i+1} + q\gamma^{i-1} \qquad i=1,\dots,N-1$$

$$\Longrightarrow p\gamma^2 - \gamma + q = 0$$

$$\Longleftrightarrow (p\gamma - q)(\gamma - 1) = 0$$

$$\Longrightarrow \gamma = q/p \quad \text{or} \quad \gamma = 1$$

If $q \neq p$, $k_i^{(1)} = \left(\frac{q}{p}\right)^i$ is a solⁿ of homog. eqtⁿ

& $k_i^{(2)} = 1$ is a solⁿ of homog. eqtⁿ

If $q/p = 1$, $k_i^{(1)} = i$ is a solⁿ of homog. eqtⁿ

& $k_i^{(2)} = 1$ as before is as well.

$\Longrightarrow$

Need to find
$k_i^{(3)}$ of non-homog. eqt^{n}:

$$k_i = 1 + p k_{i+1} + q k_{i-1}, \quad i = 1, \dots, N-1$$

Try $k_i^{(3)} = ci$,

$$\cancel{c} i = 1 + p\cancel{c}(\cancel{i}+1) + q c(\cancel{i}-1)$$
$$\Longleftrightarrow qc - pc = 1 \Longleftrightarrow c = \frac{1}{q-p} \quad \text{if } q \neq p$$

For $q = p$, try $k_i^{(3)} = ci^2$

For $q \neq p$, general sol^{n} of $Bk = 1$ is of the form

$$k_i = a\left(\frac{q}{p}\right)^i + b \cdot 1 + \frac{i}{q-p}$$

use $k_0 = 0 = k_N$ to determine a & b.

$$\Longrightarrow k_i = \frac{N}{\left(1-\left(\frac{q}{p}\right)^N\right)(q-p)}\left(\left(\frac{q}{p}\right)^i - 1\right) + \frac{i}{q-p}$$

For $p = q = \frac{1}{2}$, the gen. sol^{n} of $Bk = 1$ is of the form,

$$k_i = ai + b - i^2, \quad i = 0, 1, 2, \dots, N-1, N$$

use $k_0 = k_N = 0$ to determine a & b

$$k_i = Ni - i^2 \quad \dots$$