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LECTURE 4

WEDNESDAY April 11, 2007

M285A

NOTE TAKER:
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ASSISTANT:
ANDREW C.

Change in TA office hours
Vladimir Pesic
M 11am – noon
Tues 2:30 – 3:30 pm
AP+M 6321

Hitting Times & Hitting probabilities

$P_i(\cdot) = P(\cdot \mid X_0 = i), \quad i \in S$

Let $X = \{X_n, n \geq 0\}$
be a discrete time Markov chain with state space S

Let $A \subset S \quad A \neq \phi$ — states we care about getting to

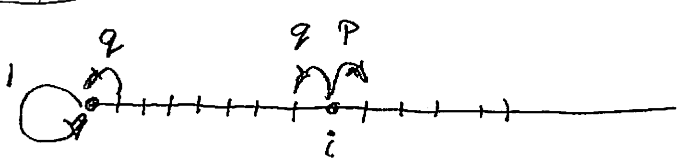
H^A = first time X is in A
$= \inf\{n \geq 0 : X_n \in A\}$ — $+\infty$ if X never gets in A

Questions of Interest – Compute

(i) $P_i(H^A < \infty)$ for $i \in S$ $=: h_i^A$
ii) $E_i[H^A]$ for $i \in S$ $=: k_i^A$

Remark
$0 \leq h_i^A \leq 1 \quad \forall i \in S$
$0 \leq k_i^A \leq \infty \quad \forall i \in S$

Example



$S = \mathbb{N}$
$p, q \geq 0$
$p + q = 1$

$A = \{0\}$

Example Continued - random walk with absorption at the origin

Rule $H_A = H^A$

First Step Analysis

Derive Equations for h_i^A, $i \in S$

If $i \in A$, $h_i^A = 1$ (since $H^A = 0$)

If $i \notin A$, condition on position of 1^{st} step

Fix $i \notin A$

$$h_i^A = P(H^A < \infty \mid X_0 = i)$$

$H^A = \inf\{n \geq 0 : X_n \in A\}$
& $\inf\{n \geq 1 : X_n \in A\}$

$$= \sum_{j \in S} P(H^A < \infty, X_1 = j \mid X_0 = i)$$

$$= \sum_{j \in S} \frac{P(H^A < \infty, X_1 = j, X_0 = i)}{P(X_0 = i)} = \sum_{j \in S} \frac{P(H_A < \infty \mid X_1 = j, X_0 = i) P(X_1 = j, X_0 = i)}{P(X_0 = i)}$$

USE MARKOV PROP

$$= \sum_{j \in S} P(H_A < \infty \mid X_1 = j, X_0 = i) P(X_1 = j \mid X_0 = i)$$

$$= \sum_{j \in S} P(\overbrace{H_A}^{\inf(n \geq 1, X_n \in A)} < \infty \mid X_1 = j) P_{ij}$$

time homogeneity

$$= \sum_{j \in S} P(H_A < \infty \mid X_0 = j) P_{ij}$$

$$= \sum_{j \in S} h_j^A P_{ij}$$

~~[illegible]~~

$h_i^A = (Ph^A)_i \quad , \quad i \notin A$

$$h^A = \begin{pmatrix} h_0^A \\ h_1^A \\ \vdots \end{pmatrix}$$

Theorem 1.3 in Norris

<u>Theorem</u>: the vector $h^A = (h_i^A : i \in S)$ is the <u>minimal</u> non negative solution of the system of linear equations

(1) $$\begin{cases} h_i^A = 1 & \text{if } i \in A \\ h_i^A = \sum_{j \in S} P_{ij} h_j^A & \text{if } i \notin A \end{cases}$$

<u>Proof</u>

Using 1st step analysis we just showed that h^A is a non-negative solution of (1)

Suppose $x = (x_i : i \in S)$ is also a non-negative solution of (1)

Want to show $x_i \geq h_i^A \ \forall i$

If $i \in A$, then $x_i = h_i^A = 1$

Consider $i \notin A$, then $x_i = \sum_{j \in S} P_{ij} x_j = \sum_{j \in A} P_{ij} x_j + \sum_{j \notin A} P_{ij} x_j$

$$= \sum_{j \in A} P_{ij} + \sum_{j \notin A} P_{ij} x_j$$

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recursion

$$\sum_{j\in A} p_{ij} + \sum_{j\notin A} p_{ij}x_j = \sum_{j\in A} p_{ij} + \sum_{j\notin A} p_{ij}\Big(\sum_{k\in A} p_{jk} + \sum_{k\notin A} p_{jk}x_k\Big) = \dots =$$

$$= \sum_{j\in A} p_{ij} + \sum_{j\notin A}\sum_{k\in A} p_{ij}p_{jk} + \dots + \sum_{j_1\notin A}\sum_{j_2\notin A}\dots\sum_{j_n\in A} p_{ij_1}p_{j_1j_2}\cdots p_{j_{n-1}j_n} + \sum_{j_1\notin A}\dots\sum_{j_n\notin A} p_{ij_1}\cdots p_{j_{n-1}j_n}x_{j_n}$$

Now, if x is non-negative, so is last term on right

So above

$$\geq \underbrace{P_i(H^A=1) + P_i(H^A=2) + \dots P_i(H^A=n)}$$

$$= P_i(H^A \leq n) \longrightarrow P_i(H^A < \infty) \quad \text{as } n\to\infty.$$

therefore $x_i \geq P_i(H^A<\infty) = h_i^A$

Now how to modify argument to get $k_i^A = E_i[H^A]$ mean hitting time

If $i\in A$, $H^A=0 \Rightarrow k_i^A = 0$

If $i\notin A$, take first step, so $H^A \geq 1$

indicator function

$$1 = \sum_{j\in S} 1_{\{X_1=j\}}$$

Fix $i\notin A$

$$k_i^A = \sum_{j\in S} E_i[H^A 1_{\{X_1=j\}}]$$

$$= \sum_{j\in S} E[H^A \cdot 1_{\{X_1=j\}} \mid X_0=i]$$

PAST — remove with MARKOV prop

$$= \sum_{j\in S} E[H^A \cdot \quad \mid X_1=j, X_0=i]\, P(X_1=j \mid X_0=i)$$

$$= \sum_{j\in S} E[H^A \cdot \quad \mid X_1=j]\, P_{ij}$$

$$= \sum_{j \in S} E[H^A \mid X_1 = j] p_{ij}$$

$$\inf\{n \geq 1 : X_n \in A\} = 1 + \inf\{n \geq 0 : X_{n+1} \in A\}$$

time homogeneity

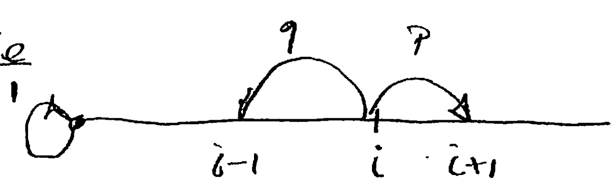
$$= \sum_{j \in S} (1 + E[H^A \mid X_0 = j]) p_{ij}$$

$$= 1 + \sum_{j \in S} p_{ij} k_j^A$$

Theorem 1.3.5 The vector of mean hitting times $k^A = (k_i^A : i \in S)$ is the minimal non-negative solution of

$$\begin{cases} k_i^A = 0 & \text{if } i \in A \\ k_i^A = 1 + \sum_{j \in S} p_{ij} k_j^A & \text{for } i \notin A \end{cases}$$

Example



$A = \{0\}$

$p = q = 1/2$

$$h_0^{\{0\}} = 1$$

$$h_i^{\{0\}} = p\, h_{i+1}^{\{0\}} + q\, h_{i-1}^{\{0\}}, \quad i \geq 1$$

$$k_0^{\{0\}} = 0$$

$$k_i^{\{0\}} = 1 + p k_{i+1}^{\{0\}} + q k_{i-1}^{\{0\}} \qquad \text{if } i \neq 0$$

$$= 1 + 1/2 \left(k_{i+1}^{\{0\}} + k_{i-1}^{\{0\}} \right) \qquad \text{for } p = q = 1/2$$

$\underline{i=1:}$ $\quad k_1^{\{0\}} = 1 + 1/2\, k_2^{\{0\}}$

$$k_2^{\{0\}} = 1 + 1/2\left(k_3^{\{0\}} + k_1^{\{0\}}\right) = 1 + 1/2\left(k_3^{\{0\}} + 1 + 1/2\, k_2^{\{0\}}\right)$$

$$= 1 + 1/2 + 1/2\, k_3^{(0)} + 1/4\, k_2^{\{0\}} \quad \geq 1 + 1/2 + 1/2\, k_3^{\{0\}}$$

DON'T ~~ADD~~ subtract $k^{\{0\}}$ ~~from~~ both SIDES unless they are not ∞

<u>END</u>