Notetaker : David Eil
Assistant Notetaker : Lucas Siga

# Lecture 3

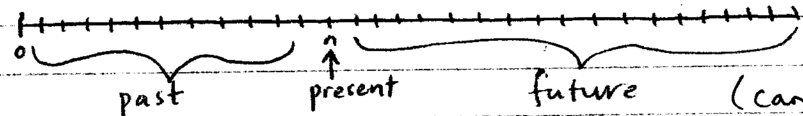
MATH 285A - 4/19/07

## Discrete Time Markov Chains, cont.

(Read: Classification of States; Hitting Times; Absorption Probabilities
Section 1.2; Section 1.3

Defn $\{X_n, n \geq 0\}$ is a Markov chain if:

$$P(X_{n+1} = i_{n+1} \mid X_0 = i_0, \ldots, X_n = i_n) = P(X_{n+1} = i_{n+1} \mid X_n = i_n) \quad \forall n, \; i_0, i_1, \ldots, i_{n+1} \in S$$




(can include present in past + future if you wish)

$= P_{i_n, i_{n+1}}$ ($P$ = transition matrix)

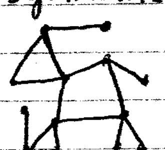
## Markov property implies:

$$P(X_{n+m} = i_{n+m}, \ldots, X_{n+1} = i_{n+1} \mid X_0 = i_0, \ldots, X_n = i_n) = P(X_{n+m} = i_{n+m}, \ldots, X_{n+1} = i_{n+1} \mid X_n = i_n)$$

$\forall n \geq 0, \; m \geq 1, \; i_0, i_1, \ldots, i_{n+m} \in S$

## Examples of Markov Chains

1. Symmetric random walk on a finite ~~path~~ graph



Graph = $(V, E)$ — $V$: vertices, $E$: edges

degree of a vertex = # of edges attached to vertex
label vertices by $i = 1, 2, \ldots, N$
$d(i)$ = degree of $i$ (assume $d(i) \geq 1$ for all $i$)
$\lambda = (\underbrace{\qquad\qquad}_{N})$ initial distribution

$X_n$ = vertex that chain is at at time $n$ (position at time $n$)
Fix $n \geq 0$, $i_0, i_1, \ldots, i_{n+1} \in S = \{1, 2, \ldots, N\}$
$P(X_{n+1} = i_{n+1} \mid X_0 = i_0, \ldots, X_n = i_n) = P(X_{n+1} = i_{n+1} \mid X_n = i_n)$,
because given a state at time $n$, we go to neighbor with prob. $= 1/d(i)$. So:

$$P(X_{n+1} = i_{n+1} \mid X_n = i_n) = \begin{cases} 0 & \text{if } i_{n+1} \text{ is not a neighbor of } i_n \\ \frac{1}{d(i_n)} & \text{if } i_{n+1} \text{ is a neighbor of } i_n \end{cases}$$

Markov because the probability of moving from one node to another is path-independent, or memoryless

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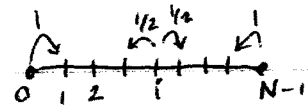



"reflecting random walk"

## 2. Branching Processes

Time $\mathbb{N} = \{0,1,2,\dots\}$ n indexes generations

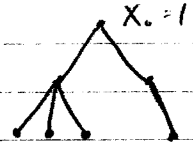
$X_n$ = # of individuals in population at time n

For each $n \geq 0$, $\{\xi_i^{n+1}\}_{i=1}^{\infty}$ are i.i.d random variables taking values in non-negative integers and $P(\xi_1^{n+1} = i) = p_i$ $i = 0,1,2,\dots$ $p_i \geq 0 \ \forall i$, $\sum_{i=0}^{\infty} p_i = 1$

$\{\xi_i^{(n+1)}\}_{i=1}^{\infty}$ are independent as n varies and independent of $X_0$

$$X_{n+1} = \sum_{i=1}^{X_n} \xi_i^{(n+1)}$$

__Ex.__ $X_0 = 1$



$X_0 = 1$

$X_1 = 2 \quad \xi_1^{(1)} = 2$

$X_2 = 4 \quad \xi_2^{(2)} = 1, \ \xi_1^{(2)} = 3$

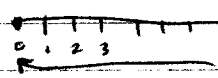
Check $\{X_n\}_{n=0}^{\infty}$ is Markov:

Fix $n \geq 0$, $i_0, i_1, \dots, i_{n+1} \in S = \mathbb{N} = \{0,1,2,\dots\}$

$P(X_{n+1} = i_{n+1} \mid X_0 = i_0, \dots, X_n = i_n) = P(\sum_{i=1}^{X_n} \xi_i^{(n+1)} = i_{n+1} \mid X_0 = i_0, \dots, X_n = i_n)$

Recall that $\{\xi_i^{(n+1)}\}_{i=1}^{\infty}$ independent of $\{\xi_i^{(1)}\}_{i=1}^{\infty}, \{\xi_i^{(2)}\}_{i=1}^{\infty}, \dots, \{\xi_i^{(n)}\}_{i=1}^{\infty}$ & $X_0$

So $P(\sum_{i=1}^{X_n} \xi_i^{(n+1)} = i_{n+1} \mid X_0 = i_0, \dots, X_n = i_n) = P(\sum_{i=1}^{i_n} \xi_i^{(n+1)} = i_{n+1}) = P(\sum_{i=1}^{X_n} \xi_i^{(n+1)} = i_{n+1} \mid X_n = i_n)$




"absorbing state" or "dead state"

## CLASSIFICATION OF STATES

We say state i leads to state j, written $i \to j$, if $P(X_n = j \mid X_0 = i) > 0$ for some $n \geq 0$ (equivalent to: $P(X_n = j$ for some $n \geq 0 \mid X_0 = i) > 0$)

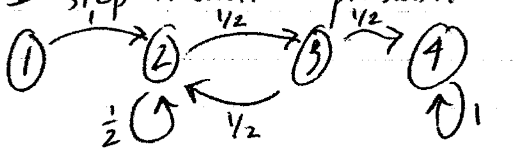
We say i and j communicate if $i \to j$ and $j \to i$, written $i \leftrightarrow j$

- communication is an equivalence relation, meaning it is:
  - (i) Reflexive ($i \leftrightarrow i$)
  - (ii) Symmetric ($i \leftrightarrow j \Rightarrow j \leftrightarrow i$)
  - (iii) ~~Transient~~/Transitive ($i \leftrightarrow j$, $j \leftrightarrow k \Rightarrow i \leftrightarrow k$)

Therefore the state space can be partitioned into <u>equivalence classes</u>. Each equivalence class contains states that communicate only with states in that class.

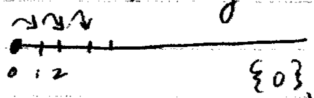
Example:

1 step transition probabilities




Communicating classes : $\{1\}, \{2,3\}, \{4\}$

Closed communicating class: A communicating class $C$ s.t. if $i \in C$ + $j \in S$ s.t. $i \to j$, then $j \in C$ ($\{4\}$, in the example)

Ex. 

, $S = \mathbb{N}$, communicating classes are $\{0\}, \{1\}, \{2\}, \ldots$ NO CLOSED COMMUNICATING CLASSES

If the state space is finite, there must be at least one closed communicating class.