BROWNIAN MOTION — LECT. 19 — 6/11/07

- Daniel Zenter
- M. Madzarevic

Limit of Random Walks (functional central limit theorem)

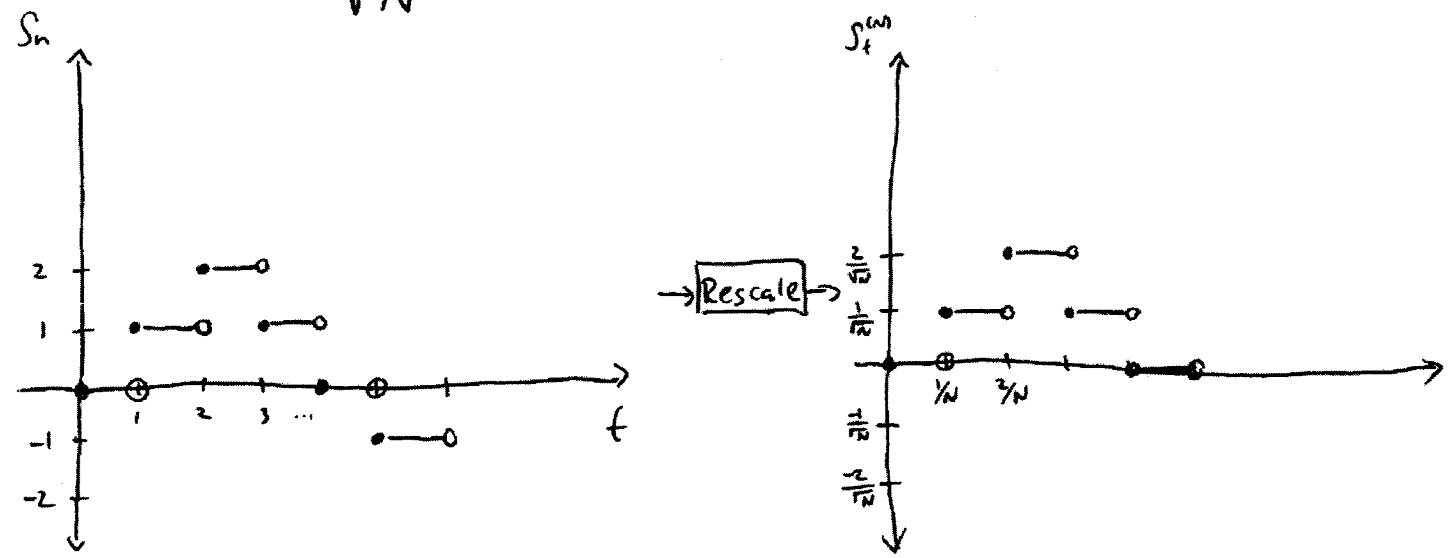
$\{\xi_i\}_{i=1}^{\infty}$ i.i.d. $P(\xi_i = +1) = P(\xi_i = -1) = \frac{1}{2}$

$S_n = \sum_{i=1}^{n} \xi_i$, $n = 0, 1, 2, \ldots$

by c.l.t.) $\frac{S_n}{\sqrt{n}} \Rightarrow X_1 \sim N(0,1)$ as $n \to \infty$

N pos integer,

$S_t^{(N)} = \frac{S_{[Nt]}}{\sqrt{N}}$, $t \geq 0$ $\quad [\cdot]$ integer part



$S^{(N)} = \{S_t^{(N)}, t \geq 0\} \underset{\text{as } N\to\infty}{\Rightarrow} \{B_t, t \geq 0\}$ 1 dim standard Brownian Motion

[convergence in distribution]

In particular, $(S_{t_1}^{(N)}, S_{t_2}^{(N)}, \ldots, S_{t_n}^{(N)}) \underset{N\to\infty}{\Rightarrow} (B_{t_1}, \ldots, B_{t_n})$ for any $0 \leq t_1 < t_2 < \cdots < t_n < \infty$

recall: $B_1 = B_1 - B_0 \sim N(0,1)$

Binomial model in finance $\longrightarrow$ Black-Scholes model

Defn A stochastic process $\{X(t), t \geq 0\}$ is a Gaussian process if for any $0 \leq t_1 < t_2 < \dots < t_n < \infty$, $(X_{t_1}, X_{t_2}, \dots, X_{t_n})$ has a multivariate normal distribution.

Characterized by giving $\mu(t) = E(X_t)$, $t \geq 0$

<u>Covariance Function:</u>

$$R(s,t) = E[(X(s) - \mu(s))(X(t) - \mu(t))], \quad 0 \leq s \leq t < \infty$$

<u>Brownian motion is a Gaussian Process</u>

Let B be a standard 1-dim Brownian motion.

Fix $0 \leq t_1 < t_2 < \dots < t_n < \infty$

$$(B_{t_1}, B_{t_2}, \dots, B_{t_n}) = \left(B_{t_1} - B_{t_0}, (B_{t_2} - B_{t_1}) - (B_{t_1} - B_{t_0}), \dots, (B_{t_n} - B_{t_{n-1}}) + \dots + (B_{t_1} - B_{t_0})\right)$$

i.e. can write $(B_{t_1}, \dots, B_{t_n})$ in terms of linear transformation of

$(B_{t_1} - B_0, B_{t_2} - B_{t_1}, \dots, B_{t_n} - B_{t_{n-1}})$ where:

$B_{t_1} - B_0, \dots, B_{t_n} - B_{t_{n-1}}$ are independent normal RV's

with means of 0 and variances $t_1, t_2 - t_1, \dots, t_n - t_{n-1}$

Hint #3 on HW 7: $P(X_2 > 0 \mid X_1 > 0)$ where X is standard 1dim BM

$(X_2, X_1) = ((X_2 - X_1) + X_1, X_1)$ where $(X_2 - X_1, X_1)$ are indep $N(0,1)$ RV's

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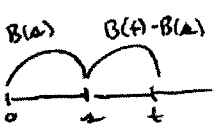
<u>For BM:</u>

$\mu(t) = E[B(t)] = 0 \qquad B(t) \sim N(0,1)$

$R(s,t) = E[B(s)B(t)] \qquad 0 \leq s \leq t < \infty$

$= E[B(s)(B(t) - B(s) + B(s))] = E[B(s)(B(t) - B(s))] + E[B(s^2)]$

$= E[\cancel{B(s)}^{0}]E[B(t) - B(s)] + s = 0 + s \implies \boxed{R(s,t) = s}$



$B(s) \sim N(0,1)$, $\quad R(s,t) = s \wedge t$, $\quad s,t \geq 0$

MARTINGALE PROPERTY

$\mathcal{F}_t = \sigma\{B_s,\ 0 \le s \le t\}$ = smallest σ-algebra containing $\sigma(B_s)$ for all $s \le t$

$\{B_t, \mathcal{F}_t, t \ge 0\}$ is a continuous time Martingale

(i) B_t is $\mathcal{F}_t$-measurable for all $t \ge 0$

(ii) $E[|B_t|] < \infty$ since $B_t \sim N(0,t)$

(iii) $E[B_t \mid \mathcal{F}_s] = B_s \quad 0 \le s < t < \infty$

<u>To Prove (iii)</u> Fix $0 \le s < t < \infty$

$$E[B_t \mid \mathcal{F}_s] = E[(B_t - B_s) + B_s \mid \mathcal{F}_s]$$

linearity $\rightarrow$
$$= E[B_t - B_s \mid \mathcal{F}_s] + \underbrace{E[B_s \mid \mathcal{F}_s]}_{\hookrightarrow B_s}$$

$B_t - B_s$ indep't of $\mathcal{F}_s$ & $B_s \in \mathcal{F}_s$ $\rightarrow$
$$= E[B_t - B_s] + B_s$$

$$= 0 + B_s = B_s \qquad B_t - B_s \sim N(0, t-s)$$

$$\boxed{\therefore E[B_t \mid \mathcal{F}_s] = B_s}$$

<u>Exercise</u>

$\{B_t^2 - t, \mathcal{F}_t, t \ge 0\}$ is a Martingale

(show $E[B_t^2 - t \mid \mathcal{F}_s] = B_s^2 - \overset{s}{\cancel{t}}$ for $0 \le s < t < \infty$)

<u>Thm:</u> <u>If</u> $X = \{X_t, t \ge 0\}$ is a 1-dim cts stochastic process s.t. $X_0 = 0$, $\{X_t, \mathcal{F}_t, t \ge 0\}$ is a martingale & $\{X_t^2 - t, \mathcal{F}_t, t \ge 0\}$ is a martingale where $\mathcal{F}_t = \sigma\{X_s : 0 \le s \le t\}$, $t \ge 0$

<u>then</u> X is a standard 1-dim Brownian Motion

MARKOV PROPERTY

$P(B_t \in A \mid B_u : 0 \le u \le s)$ for $0 \le s < t < \infty$

$= P(B_t \in A \mid B_s)$ for any measurable set A in $\mathbb{R}$

In fact, enough to show

$P(B_t \in A \mid B_{u_1} \in A_1, \ldots, B_{u_n} \in A_n, B_s \in A')$ for $0 \le u_1 < u_2 < \cdots < u_n < s < t$

$= P(B_t \in A \mid B_s \in A')$ for any meas. sets $A_1, \ldots, A_n, A', A$

LHS $= \boxed{P(B_t \in A \mid B_{u_1} \in A_1, \ldots, B_{u_n} \in A_n, B_s \in A')}$

$= P(B_t - B_s \in A - B_s \mid B_{u_1} \in A_1, \ldots, B_{u_n} \in A_n, B_s \in A')$

$= P(B_t - B_s \in A - B_s \mid B_s \in A')$

$= \boxed{P(B_t \in A \mid B_s \in A')}$