# MATH 285 Lecture Notes

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June 6, 2007

**Optional Stopping**

Given a filtration, $\{F_n, n = 0, 1, 2, ...\}$, a stopping time (optional time) (relative to the filtration) is a random variable.
T: $\Omega \longrightarrow \{0, 1, 2, ...\} \cup \{\infty\}$
such that for each $n = 0, 1, 2, ..., \{T = n\} = \{\omega \in \Omega : T(\omega) = n\} \in F_n$.

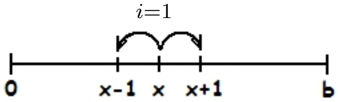
**Simple Stopping Theorem**

Suppose $M = \{M_n, F_n, n = 0, 1, 2, ...\}$ is a martingale and $T$ is the stopping time relative to $\{F_n\}_{n=0}^{\infty}$.
Assume there is a constant $K$ such that $T \leq K$ a.s. then $E[M_T] = E[M_0]$.
$E[M_n] = E[M_0]$ for each n.
$(M_T)(\omega) = (M_{T(\omega)})(\omega)$

Coin Flipping Example
If $\omega = TTTHT$, then $\tau = \inf\{n \geq 0 : x_n = H\}$

**Example**

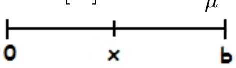
$\{\xi_i\}_{i=1}^{\infty}$ iid $P(\xi_i = +1) = p$, $P(\xi_i = -1) = q$ where
$p + q = 1$, $0 < p < 1$, $p \neq q$
$\mu = E[\xi_i] = p - q \neq 0$
$X_n = x + \sum_{i=1}^{n} \xi_i$ where $x$ is an integer between 0 and $b$ ($b > 0$ integer)




$M_n = X_n - n\mu, n = 0, 1, 2, ...$
$F_n = \sigma\{X_1, ..., X_n\}, n = 1, 2, ...$
$F_0 = \{\emptyset, \Omega\} = \sigma\{X_0\}$
Claim

$\{M_n, F_n, n = 0, 1, 2, ...\}$ is a martingale.
$T = \inf\{n \geq 0 : X_n \text{ or } b\}$

T is a stopping time
Fix $n \in \{0, 1, 2, ...\}$
$\{T = n\} = \{X_0 \notin \{0, b\}, \{X_1 \notin \{0, b\}, ..., X_{n-1} \notin \{0, b\}, X_n \in \{0, b\}\} \in F_n$
**Theorem**
Suppose $S$ and $T$ are two stopping times. Then $S \wedge T$ is also a stopping time. Also any deterministic time is a stopping time. Thus $T \wedge N$ is a stopping time for each $N \in \{0, 1, 2, ...\}$.
So $T \wedge N \subseteq N$ and can apply stopping theorem.
$E[M_{T \wedge N}] = E[M_0]$
$M_n = X_n - n\mu$ and $M_0 = X_0 = x$
$\implies E[X_{T \wedge N} - (T \wedge N)\mu] = x$
$|X_{T \wedge N}| \leq b$, then $E[X_{T \wedge N}] - E[(T \wedge N)\mu] = x$
$\mu E[T \wedge N] = E[X_{T \wedge N}] - x$
$(\star)$ $E[T \wedge N] = \frac{E[X_{T \wedge N}] - x}{\mu}$
$|\frac{E[X_{T \wedge N}] - x}{\mu}| \leq \frac{b + x}{|\mu|}$
$E[T \wedge N] \leq \frac{b + x}{|\mu|}$
Let $N \to \infty$, by Monotone Convergence.
$E[T] = E[\lim_{N \to \infty} T \wedge N]$
By Monotone Convergence Theorem
$$E[T] = \lim_{N \to \infty} E[T \wedge N] \leq \frac{b + x}{|\mu|} < \infty$$
$\implies T < \infty$ a.s. and in fact $E[T] < \infty$
Let $N \to \infty$ in $(\star)$
$E[T \wedge N] = \frac{E[X_{T \wedge N}] - x}{\mu}$
As $N \to \infty$ by Monotone Convergence
$E[T \wedge N] \to E[T]$
$(T \wedge N)(\omega) \to T(\omega)$ as $N \to \infty$ for a.e. $\omega$
$(X_{T \wedge N})(\omega) \to X_T(\omega)$
As $N \to \infty$ by Bounded Convergence
$\frac{E[X_{t \wedge N}] - x}{\mu} \to \frac{E[X_T] - x}{\mu}$
$\implies E[T] = \frac{E[X_T] - x}{\mu}$ for a.e. $\omega$ $(X_{T \wedge N})(w) \to (X_T)(\omega)$ bounded convergence.
so $E[T] = \frac{b P_x(x_T = b) - x}{\mu}$

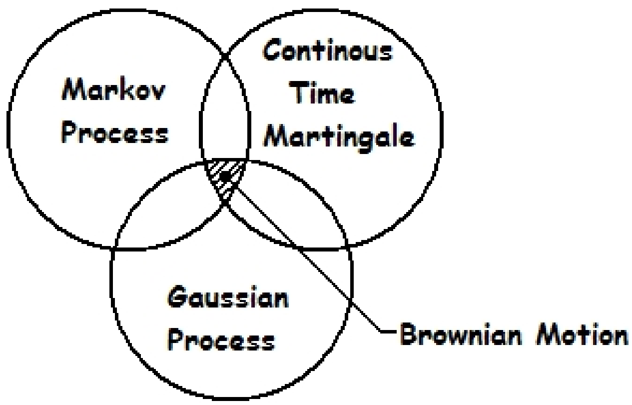



[ Martingale $Q_n = (\frac{q}{p})^{x_n}$] when T is finite w.p. 1 gives $P(x_T = b)$

**Brownian Motion** - continuous time, continuos state (Wiener Process).

$S = \Re$(one dimensional)

**Stochastic Process** $\{B_t, t \geq 0\}$

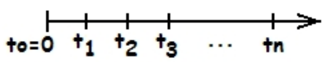



**Definition**

A standard one dimensional Brownian Motion is a stochastic process $\{B_t, t \geq 0\}$ taking value in $\Re$, such that

(i) $B_0 = 0$ a.s.

(ii) (independent incerements)



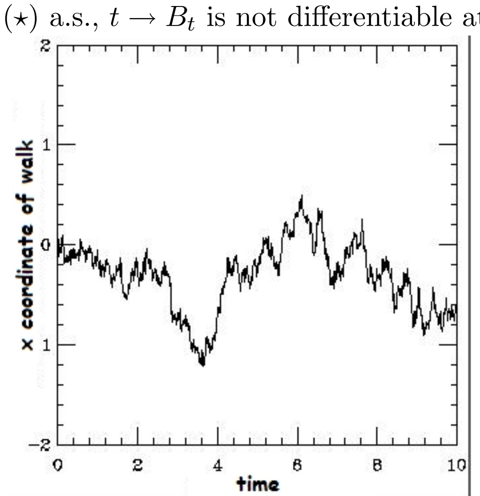

$\{B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, ..., B_{t_n} - B_{t_{n-1}}\}$ are independent for any $0 = t_0 < t_1 < ... < t_n$ and any $n = 1, 2, ...$

(iii) For any s, t, $0 \leq s < t < \infty$

$B_t - B_s$ is a normal r.v. with mean 0 and variance $t - s$ (stationary increment)

(iv) a.s., $t \to B_t$ is continuous

($\star$) a.s., $t \to B_t$ is not differentiable at any $t$.



Evidence for $(\star)$

Fix $t > 0$

By distribution, $\frac{B_{t+n} - B_t}{n} = \frac{B_h}{h} = \frac{\sqrt{h} B_1}{h} = \frac{B_1}{\sqrt{h}} \to \pm\infty$ as $h \to 0$

"$\frac{dB_t}{dt}$" $=$ white noise

$\int f(t) dB_t = \int f(t)$"$\frac{dB_t}{dt}$"$dt$ does not exist.