6/4/07 ①

MATH 285

Note-taker: Feliks Dushatsky

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Kumar.

Lecture 17

1) Make-up class

Next Monday, June 11, 5-6:20pm
AP+M 6402

2) Office Hours: (AP+M 6121) Prof. Williams

Wed. June 6 : 3-3:30pm
Friday June 8 : 3-4 pm
Monday June 11 : 3-4pm

Martingales

$\{M_n, \mathcal{F}_n, n=0,1,2,\ldots\}$ discrete time stochastic process.

(i) for each n, M_n is $\mathcal{F}_n$-measurable
(write: $M_n \in \mathcal{F}_n$)

(ii) for each n, $E[|M_n|] < \infty$

(iii) $E[M_{n+1} \mid \mathcal{F}_n] = M_n$ for each n, Consequence of martingale property:

$$E[M_n \mid \mathcal{F}_m] = M_m \qquad m < n.$$

(Excercise)

example: Branching Process

Time: generations $n = 0, 1, 2, \ldots$

$X_n = \#$ individuals in generation n

$X_0 = 1$

$$X_{n+1} = \sum_{i=1}^{X_n} \xi_i^{(n+1)}$$

$\xi_i^{(n+1)}$ = (random) # of offspring of i^{th} individual in generation n produced for generation $n+1$.

$\{ \xi_i^{(m)}, i = 1, 2, \ldots \}_{m=1}^{\infty}$ iid taking values in set $\{0, 1, 2, \ldots\}$ with finite mean $\mu = E[\xi_1^{(1)}] < \infty$

$P(\xi_1^{(1)} = 1) \neq 1$ (non-degeneraly)

Martingale:

Filtration:

$$\mathcal{F}_n = \sigma\{X_0, X_1, \ldots, X_n\}$$

$$n = 0, 1, 2, \ldots$$

$$M_n = X_n / \mu^n, \quad n = 0, 1, 2, \ldots$$

Check $\{M_n, \mathcal{F}_n, n = 0, 1, 2, \ldots\}$ is a martingale.

(i) M_n is $\mathcal{F}_n$-measurable because X_n is trivially $\mathcal{F}_n$-measurable + μ^n is just a constant, $n = 0, 1, 2, \ldots$

(ii) $M_0 = X_0/\mu^0 = 1$ is clearly integrable. Prove by induction that M_n is integrable for each n. Suppose M_n integrable, i.e. $E[M_n] < \infty$. Want to prove M_{n+1} is integrable.

$$E[M_{n+1}]$$

$$= \frac{1}{\mu^{n+1}} E[X_{n+1}]$$

$$= \frac{1}{\mu^{n+1}} E\left[\sum_{i=1}^{X_n} \xi_i^{(n+1)}\right]$$

$$= \frac{1}{\mu^{n+1}} E\left[E\left[\sum_{i=1}^{X_n} \xi_i^{(n+1)} \,\middle|\, \mathcal{F}_n\right]\right]$$

$$= \frac{1}{\mu^{n+1}} E\left[X_n \cdot E\left[\xi_1^{(n+1)}\right]\right]$$

$$= \frac{1}{\mu^{n}} E\left[X_n\right]$$

$$= E\left[X_n/\mu^n\right]$$

$$= E\left[M_n\right] < \infty$$

Tower Property

$H \subset G$

$E[E[X|G]|H] = E[X|H]$

$H = \mathcal{F}_0$

(iii) Fix $n \in \{0, 1, 2, \ldots\}$

$$E[M_{n+1} | \mathcal{F}_n] = \frac{1}{\mu^{n+1}} E[X_{n+1} | \mathcal{F}_n]$$

$$= \frac{1}{\mu^{n+1}} E\left[\sum_{i=1}^{X_n} \xi_i^{(n+1)} \,\middle|\, \mathcal{F}_n\right]$$

$$= \frac{1}{\mu^{n+1}} X_n \cdot E\left[\xi_1^{(n+1)}\right]$$

$$= \frac{X_n}{\mu^n} = M_n$$

X_n is $\mathcal{F}_n$-meas.
$\{\xi_i^{(n+1)}\}_{i=1}^{\infty}$ is indep. of $\{\mathcal{F}_n\}$

Since $\{M_n\}_{n=0}^{\infty}$ is a martingale,

$$E[M_n] = E[M_0] = 1$$

$$\Rightarrow E\left[X_n / \mu^n\right] = 1 \quad \text{for all } n$$

$$\Rightarrow E[X_n] = \mu^n \quad \text{for all } n$$

$$\Rightarrow \lim_{n\to\infty} E[X_n] = \begin{cases} 0, & \mu < 1 \\ 1, & \mu = 1 \\ +\infty, & \mu > 1 \end{cases}$$

Fatou's Lemma

$$\Rightarrow E\left[\underline{\lim}_{n\to\infty} X_n\right] \le \underline{\lim}_{n\to\infty} E[X_n]$$

$$\underline{\lim}_{n\to\infty} = \lim_{n\to\infty} \inf X_n = \sup_m \inf_{n \ge m} X_n$$

When $\mu < 1$,

$$\Rightarrow E\left[\underline{\lim}_{n\to\infty} X_n\right] = 0$$

$$\Rightarrow \underline{\lim}_{n\to\infty} X_n = 0 \quad \text{a.s.} \quad (\text{since } X_n \ge 0)$$

$$\Rightarrow \lim_{n\to\infty} X_n = 0 \quad \text{a.s.}$$

If $\mu < 1$, population dies out with probability 1.

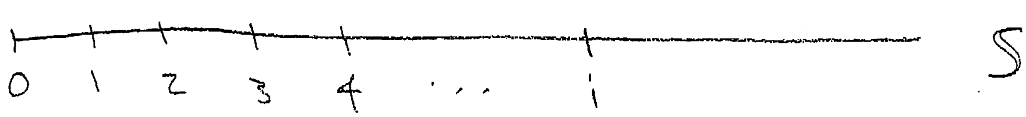
Consider $\mu \geq 1$ from now on.

$\{X_n\}_{n=0}^{\infty}$ is a Markov Chain,

$$P(X_{n+1} = j \mid X_0, X_1, \ldots, X_n)$$

$$= P\left(\sum_{i=1}^{X_n} \xi_i^{(n+1)} = j \mid X_0, X_1, \ldots, X_n\right)$$

$$= P\left(\sum_{i=1}^{X_n} \xi_i^{(n+1)} = j \mid X_n\right)$$ since $\{\xi_i^{(n+1)}\}_{i=1}^{\infty}$ are indep. of $X_0, \ldots, X_n$



Case 1: $\alpha = P(\xi_i^{(n+1)} = 0) > 0$

From any state i, there is a positive probability α^i that the MC $\{X_n\}$ jumps to zero from there. State $\{0\}$ is absorbing.

So, every state $i \geq 1$ is transient.

Case 2: $P(\xi_i^{(n+1)} = 0) = 0$

Since $P(\xi_i^{(n+1)} = 1) \neq 1$, $P(\xi_i^{(n+1)} \geq 2) > 0$

From any state i, cannot reach states less than i (no individual can have less than one offspring) and there is a positive probability of going to a state greater than i. Every state $i \geq 1$ is transient

↳ So $X_n \to \infty$ as $n \to \infty$ a.s.

Consider Case 1 only.

$\mu \geq 1$

probability generating function for $\xi_1^{(1)}$:

$$\phi(t) = E\left[t^{\xi_1^{(1)}}\right] = \sum_{k=0}^{\infty} t^k \cdot P(\xi_1^{(1)} = k),$$

$0 \leq t \leq 1$

$\phi'(1) = \mu$

Claim

$$E[t^{X_n}] = \underbrace{(\phi \circ \phi \circ \cdots \circ \phi)}_{n\text{-times}}(t), \quad n = 0, 1, 2, \ldots$$

$$= \phi^{(n)}(t)$$

ex. $E[t^{X_2}] = \phi(\phi(t))$

n=0 $E[t^{X_0}] = t$

n=1 $E[t^{X_1}] = \phi(t)$

t=0

$\phi^{(n)}(0) = P(X_n = 0)$

What is $\lim_{n\to\infty} P(X_n = 0)$

$\qquad \| $

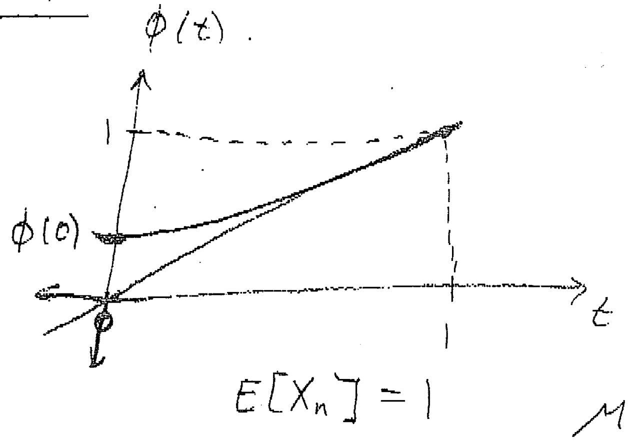
$P(X_n = 0 \text{ for some } n) = q$

$$q = \lim_{n\to\infty} P(X_n = 0) = \lim_{n\to\infty} \phi^{(n)}(0) = \lim_{n\to\infty} \phi^{(n+1)}(0)$$

$$= \lim_{n\to\infty} \phi(\phi^{(n)}(0)) = \phi(\lim_{n\to\infty} \phi^{(n)}(0)) = \phi(q)$$

(ϕ is continuous

$\mu = 1$



$\phi(1) = 1$

$\phi(0) = P(X_1 = 0)$ (Case 1)

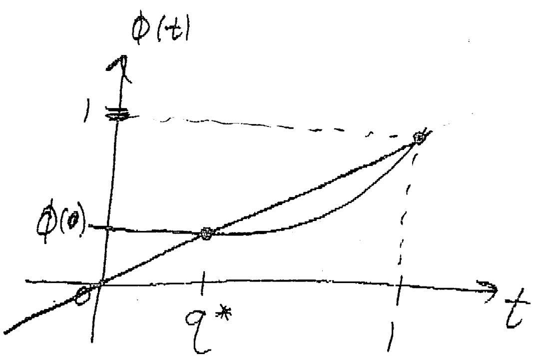
$P(\xi_1^{(1)} = 0) > 0$

$\phi(t) = E[t^{\xi_1^{(1)}}]$

$\mu = 1$

only one fixed point $\Rightarrow q = 1$

$\mu > 1$



$\phi'(1) = \mu$

$q = \lim_{n \to \infty} \phi^{(n)}(0)$

$\mu > 1$

$q^* = q < 1$

So for $\mu \le 1$, $P(X_n = 0 \text{ for some } n) = 1$ + for $\mu > 1$, $P(X_n = 0 \text{ for some } n) < 1$.

Previous Claim

$$E[t^{X_n}] = \underbrace{(\phi \circ \phi \circ \cdots \circ \phi)}_{n\text{-times}}(t), \quad n = 0, 1, 2, \ldots$$

$$= \phi^{(n)}(t)$$

proof by induction

$n=1$ $E[t^{X_1}] = E[t^{\xi_1^{(1)}}] = \phi(t) = \phi^{(1)}(t)$

Suppose true for some $n \geq 1$.

$$E\left[t^{X_{n+1}}\right]$$

$$= E\left[t^{\sum_{i=1}^{X_n} \xi_i^{(n+1)}}\right]$$

$$= \sum_{l=0}^{\infty} E\left[t^{\sum_{i=1}^{l} \xi_i^{(n+1)}} \,\middle|\, X_n = l\right] \cdot P(X_n = l)$$

$$= \sum_{l=0}^{\infty} E\left[t^{\sum_{i=1}^{l} \xi_i^{(n+1)}}\right] \cdot P(X_n = l)$$

$$= \sum_{l=0}^{\infty} \phi(t)^l \cdot P(X_n = l)$$

$$= E\left[\phi(t)^{X_n}\right]$$

$$= \underbrace{(\phi \circ \dots \circ \phi)}_{n\text{-times}}(\phi(t)) = \phi^{(n+1)}(t)$$