Lecture 16

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MATH 285 NOTES

Conditional Expectation + Martingales (Discrete Time)

$(\Omega, \mathcal{F}, P)$ probability space.

$\mathcal{G}$ - sub-σ-algebra of $\mathcal{F}$
X - random variable, $E[|X|] < \infty$

Conditional expectation of X given $\mathcal{G}$ is a random variable Y
(i) Y is $\mathcal{G}$-measurable
(ii) $E[Y 1_A] = E[X 1_A]$ for all $A \in \mathcal{G}$

Example: often $\mathcal{G}$ is generated by finitely many random variables $X_0, X_1, X_2, \ldots, X_n$

Given a random variable Z, the σ-algebra generated by Z is smallest σ-algebra containing all sets of the form
$\{\omega \in \Omega : Z(\omega) \in (a, b]\}$
for $-\infty < a < b < \infty$

Denote this σ-algebra by $\sigma(Z)$

For r.v.'s $X_0, X_1, \ldots, X_n$, the σ-algebra generated by them is the smallest σ-algebra that contains $\sigma(X_0), \sigma(X_1), \ldots, \sigma(X_n)$

If $\mathcal{G} = \sigma(X_0, X_1, \ldots, X_n)$ (σ-algebra generated by $X_0, X_1, \ldots, X_n$) then

$$E[X|\mathcal{G}] = \phi(X_0, X_1, \ldots, X_n)$$

where $\phi : \mathbb{R}^{n+1} \to \mathbb{R}$ is measurable

Special Case

$X_0, X_1, \ldots, X_n$ take values in a discrete (finite or countable) set

Fix values $i_0, i_1, \ldots, i_n$ for $X_0, X_1, \ldots, X_n$

$A = \{X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n\}$
$= \{\omega \in \Omega : X_0(\omega) = i_0, \ldots, X_n(\omega) = i_n\}$ (Assume $P(A) \neq 0$)

Seeking Y:
$\mathcal{G}$-measurable $\Rightarrow$ Y must be constant on A

$E[Y 1_A] = E[X 1_A]$

$\Rightarrow$ (value of Y on A) $P(A) = E[X 1_A]$
$\Rightarrow$ value of Y on A $= E[X 1_A]/P(A)$

$$\underbrace{E[X \mid \overbrace{\sigma(X_0, X_1, \ldots, X_n)}^{\mathcal{G}}]}_{=Y} \triangleq E[X \mid X_0, X_1, \ldots, X_n]$$

$$E[X \mid X_0, X_1, \ldots, X_n] = \frac{E[X 1_{\{X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n\}}]}{P(X_0 = i_0, \ldots, X_n = i_n)} \text{ on } A$$

Properties

Assume $\mathcal{G}$ sub-algebra of $\mathcal{F}$ + $E[|X|] < \infty$
(i): if X is $\mathcal{G}$-measurable, $E[X \mid \mathcal{G}] = X$
(ii) if X is independent of $\mathcal{G}$, then $E[X \mid \mathcal{G}] = E[X]$

X is indept of $\mathcal{G}$

iff $P(\{X \in B\} \cap G) = P(X \in B) \cdot P(G)$
for all $B = (a, b]$, $-\infty < a < b < \infty$, $G \in \mathcal{G}$

(iii) if $a_1, a_2 \in \mathbb{R}$,

$$E[a_1 X + a_2 \mid \mathcal{G}] = a_1 E[X \mid \mathcal{G}] + a_2$$

(iv) Tower Property: Suppose $\mathcal{H}$ is a σ-algebra, + $\mathcal{H} \subset \mathcal{G}$

$$E[E[X \mid \mathcal{H}] \mid \mathcal{G}] = E[E[X \mid \mathcal{G}] \mid \mathcal{H}]$$

$$= E[X \mid \mathcal{H}]$$

(v) Suppose Z is a $\mathcal{G}$-measurable, and $E[|XZ|] < \infty$,

Then $E[XZ \mid \mathcal{G}] = Z\, E[X \mid \mathcal{G}]$

(vi) if $X \le W$ then $E[X \mid \mathcal{G}] \le E[W \mid \mathcal{G}]$ (assuming $E[|W|] < \infty$)

(vii) $|E[X \mid \mathcal{G}]| \le E[|X| \mid \mathcal{G}]$ a.s.

(viii) Monotone convergence

if $\{X_n\}_{n=1}^{\infty}$ is a sequence of r.v.'s and $X_n \geq 0$,
+ $X_n \uparrow X$ a.s. as $n \to \infty$
then $E[X_n \mid \mathcal{G}] \uparrow E[X \mid \mathcal{G}]$ a.s. as $n \to \infty$

(ix) Jensen's Inequality

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be convex and $E[|\varphi(X)|] < \infty$

Then $\varphi(E[X \mid \mathcal{G}]) \le E[\varphi(X \mid \mathcal{G})]$ a.s

MARTINGALES (discrete time)

$\mathbb{T} = \{0, 1, 2, \dots\}$

Filtration: Increasing Sequence of sub-σ-algebras of $\mathcal{F}$ ie. $\{\mathcal{F}_n, n = 0, 1, 2, \dots\}$ where each $\mathcal{F}_n$ is a sub-σ-algebra of $\mathcal{F}$ + $\mathcal{F}_n \subset \mathcal{F}_{n+1}$, $n = 0, 1, 2, \dots$

Example:

Flip a coin infinitely many times, let
X_n = outcome of n^{th} coin flip
(0 if tail, 1 if head)
$\mathcal{F}_n = \sigma(X_1, \dots, X_n)$, $n = 1, 2, \dots$
$\mathcal{F}_0 = \{\emptyset, \Omega\}$

Defn: A martingale (relative to the filtration $\{\mathcal{F}_n\}_{n=0}^{\infty}$) is a real-valued stochastic process $\{M_n, n = 0, 1, 2, \dots\}$ such that

(i) M_n is an $\mathcal{F}_n$-measurable r.v. for each $n = 0, 1, 2, \dots$
(ii) $E[|M_n|] < \infty$ for $n = 0, 1, 2, \dots$
* (iii) $E[M_{n+1} \mid \mathcal{F}_n] = M_n$, $n = 0, 1, 2, \dots$
("fair game")

A (sub/super) martingale is defined similarly but with $(\geq / \leq)$ in place of $=$ in (iii).

Property: $E[M_n] = E[M_0]$ for all $n \geq 0$

Illustration of idea ~~of proof~~:

$$E[M_1 | \mathcal{F}_0] = M_0$$

$$\Rightarrow E[E[M_1 | \mathcal{F}]] = E[M_0]$$

Tower property $\Rightarrow E[M_1] = E[M_0]$

Example: coin flipping

Fair Coin, independent coin flips

$$X_i = \begin{cases} +1 & \text{if } i^{th} \text{ coin flip is heads} \\ -1 & \text{if } i^{th} \text{ coin flip is tails} \end{cases}$$

$P(X_i = +1) = P(X_i = -1) = \frac{1}{2}$

$\mathcal{F}_n = \sigma\{X_1, \ldots, X_n\}$

Wealth at time n $= W_0 + \sum_{i=1}^{n} X_i$, $n = 0, 1, 2, \ldots$

$= W_n$

$\{W_n, n = 0, 1, 2, \ldots\}$ is a martingale relative to $\{\mathcal{F}_n\}_{n=0}^{\infty}$

(i) $W_n = W_0 + \sum_{i=1}^{n} X_i$ $\quad W_0 \equiv$ constant > 0
is $\mathcal{F}_n$-measurable being a function of $X_1, X_2, \ldots, X_n$
$n = 0, 1, 2, \ldots$ (continuous)

(ii) $E[|W_n|] \leq W_0 + n < \infty$ for $n = 0, 1, 2, \ldots$

(iii) $E[W_{n+1} | \mathcal{F}_n] = E[W_0 + \sum_{i=1}^{n+1} X_i | \mathcal{F}_n] = E[W_n + X_{n+1} | \mathcal{F}_n]$

$= E[W_n | \mathcal{F}_n] + E[X_{n+1} | \mathcal{F}_n]$

$= W_n + E[X_{n+1}] = W_n + 0$

$= W_n$ ✓