Lecture 15

Math 285B
Williams
Class Notes

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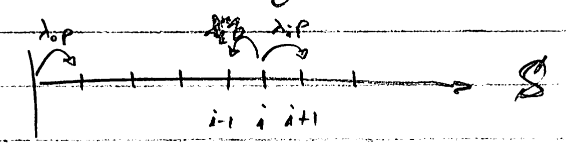
<u>Example from Monday</u>



Discrete time skeleton: recurrent iff $p \le q \implies$ CTS time MC does not explode if $p \le q$.

Also from last time: By using detailed balance

$$\pi_i = \pi_0 \left(\frac{\lambda_0}{\lambda_i}\right)\left(\frac{p}{q}\right)^i \qquad i = 0, 1, \dots$$

this will be a stationary distribution if

$$\sum_i \frac{1}{\lambda_i}\left(\frac{p}{q}\right)^i < \infty.$$

<u>Case 2</u>: Assume, $\lambda_i = 2^i$, $i = 0, 1, 2 \dots$

$$\therefore \pi_i = \pi_0 \left(\frac{p}{2q}\right)^i \quad i = 0, 1, 2 \dots \qquad (\lambda_0 = 1)$$

We know that $\sum_i \left(\frac{p}{2q}\right)^i < \infty$ iff $\frac{p}{2q} < 1$

From discrete-time MC, we are gauranteed non-explosion if $\frac{p}{q} \le 1$.

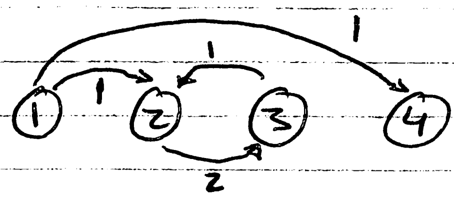
For $\frac{p}{q} \le 1$, we have non-explosion & existence of stationary distribution π. By basic limit thm this implies pos. recurrence <u>and</u> π will be a steady state distribution.

(note: discrete time skeleton is null recurrent for $p = q$)

For $1 < \frac{p}{q} < 2$: discrete time skeleton is transient and so cts time MC is also transient ($\therefore$ not recurrent nor pos. recurrent)
(In fact, cts time MC can explode when $1 < \frac{p}{q} < 2$).

Another Example:

$$Q = \begin{pmatrix} -2 & 1 & 0 & 1 \\ 0 & -2 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$



Communicating Classes: $\{1\}, \{2,3\}, \{4\}$

notes: no arrows leading into 1.
no arrows out of set $\{2,3\}$

$\{1\}$ – pos. probability of leaving and then once you leave you never come back. $\therefore$ Transient.

$\{2,3\}$ – no arrows out of $\{2,3\}$. $\therefore$ Not Transient, hence recurrent (finitely many states $\Rightarrow$ pos. recurrent.)

$\{4\}$ – clearly recurrent, since starting there means staying there forever.

Can determine stationary distribution for each pos. recurrent communicating class:

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$\{4\}$: $\pi_4^{\{4\}} = 1$ $\quad \left(\pi_i^{\{4\}} = 0 \quad i = 1, 2, 3\right)$

$\{2,3\}$: $\pi_2^{\{2,3\}}$, $\pi_3^{\{2,3\}}$ non-zero $\left(\pi_i^{\{2,3\}} = 0, \quad i = 1, 4\right)$

Example (cont'd):

$$-2\pi_2^{\{2,3\}} + 1\,\pi_3^{\{2,3\}} = 0$$
$$2\pi_2^{\{2,3\}} - 1\,\pi_3^{\{2,3\}} = 0$$
$$\Longrightarrow \quad \pi_3^{\{2,3\}} = 2\pi_2^{\{2,3\}}$$
$$\pi_2^{\{2,3\}} + \pi_3^{\{2,3\}} = 1$$

$$\therefore \quad \boxed{\pi_2^{\{2,3\}} = \frac{1}{3} \;;\; \pi_3^{\{2,3\}} = \frac{2}{3}}$$

So, what happens to "poor" state 1?

If we start in 2,3,4 can determine
$\lim_{t\to\infty} P_{ij}(t)$ for any i,j

If we start in state 1,

$$P_{1j}(t) = P\left(X(t)=j \mid X(0)=1,\ J_1 \le t,\ X(J_1) \text{ being in same communicating class as } j\right)$$
$$\times\ P\left(J_1 \le t,\ X(J_1) \text{ is in same cc as } j \mid X(0)=1\right).$$

where $J_1 = \inf\{t \ge 0 : X(t) \ne 1\}$ and $\underline{j \ne 1}$.

By the strong Markov Property for X:

$$P_{1j}(t) = P\left(X(t-J_1) = j \mid X(0) = \text{place where } X \text{ jumped to @ time } J_1 \text{ is in same c.c. class as } j\right)$$
$$\times\ P\left(J_1 \le t \mid X(0)=1\right) \cdot P\left(\text{jump from 1 to cc containing } j \text{ when jump}\right)$$

Example (cont'd):

So, what happens when we take limit as $t \to \infty$?
(assume $j = 3$).

$$\lim_{t \to \infty} P_{13}(t) = \lim_{t \to \infty} P(X(t-J_1) = 3 \mid X(0) = 2)(P_{12})$$
$$= \pi_3^{\{2,3\}} \cdot \frac{1}{2}$$

This concludes cts MC class coverage!

Conditional Expectation:

Let $(\Omega, \mathcal{F}, P)$ be a probability space.

$\Omega \to$ set of possible outcomes
$\mathcal{F} \to$ σ-algebra (events)
$P \to$ Probability measure on our space.

Let X be a random variable: $X : \Omega \to \mathbb{R}$.
(for any interval (a,b): $-\infty < a < b < \infty$,
$[\omega : X(\omega) \in (a,b)]$ is an event, i.e. an element of $\mathcal{F}$).

Assume $E[|X|] < \infty$ (here E is expected value under P).

Let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$.
(a collection of events in $\mathcal{F}$ that is closed under countable unions & complements).
+ contains ϕ + Ω.)

<u>Def'n</u>: The <u>conditional expectation</u> of X given $\mathcal{G}$ is a random variable Y such that

(i.) Y is $\mathcal{G}$-measurable

$\left(\text{i.e. for any } -\infty < a < b < \infty \quad \{ \omega \in \Omega : Y(\omega) \in (a,b) \} \in \mathcal{G} \right)$

(ii.) for any $A \in \mathcal{G}$

$$E[Y \cdot 1_A] = E[X \cdot 1_A]$$

Luckily, there is such a conditional expectation and it is unique up to a.s. equivalence.

(i.e. <u>if</u> $Y + Y'$ are two such conditional expectations <u>then</u> $P(Y = Y') = 1$.)

<u>Example</u>: Coin Flipping.

Ω = outcomes of two coin flips
$= \{ (HH), (TT), (HT), (TH) \}$.

$\Omega =$

HH	HT
TH	TT

$\mathcal{F}$ = all possible subsets of Ω (ϕ, Ω are events)

For fair coins, independent:

$$P(HH) = P(TT) = P(HT) = P(TH) = \frac{1}{4}$$

$\mathcal{G}$ corresponds to information associated with seeing the first coin flip. outcome.

Convienently we have a partition, $\mathcal{P} = \{ \{HH, HT\}, \{TH, TT\} \}$

- $\mathcal{G}$ is generated by taking all unions of sets in $\mathcal{P}$ $\therefore \mathcal{G} = \{ \phi, \Omega, \{HH, HT\}, \{TH, TT\} \}$.

Example (cont'd):

$X =$ outcome of second coin flip : $\begin{cases} \text{if heads gives } 1 \\ \text{if tails gives } 0 \end{cases}$

Notation: $E[X \mid \mathcal{G}]$ used to denote conditional expected value of X given $\mathcal{G}$.

Compute $Y = E[X \mid \mathcal{G}]$ $\longrightarrow$ will be constant on each set in partition $\mathcal{P}$

Fix $A = \{HH, HT\} \in \mathcal{P}$

Want $E[Y 1_A] = E[X 1_A]$ (1)

Let $y_1 =$ value of Y on $\{HH, HT\}$,

$$(1) \Rightarrow y_1 P(A) = E[X 1_A]$$

$$\Rightarrow y_1 = \frac{E[X 1_A]}{P(A)} = \frac{1 \cdot \frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$$

Similarly, $A_2 = \{(TH), (TT)\} \Rightarrow Y\ y = \frac{1}{2}$ on A_2