MATH 285A Stochastic Processes (05/21/2007) Lecture 14

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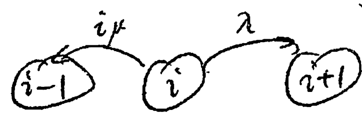
Continuous Time Markov Chains

Last time E.g. Biochemical Reaction.

~~State~~: $X(t)$ = # of molecule A present at time t



External Production: $\phi \xrightarrow{\lambda} A$

Decay Rate : $A \xrightarrow{\mu} \phi$

Poisson Property

Another Representation of X:

Let N_1, N_2 be independent Poisson processes of rate 1 each.

Then $X(t) = X(0) + N_1(\lambda t) - N_2\left(\int_0^t \mu X(s)\,ds\right)$

$$P[X(t+h) - X(t) = 1 \mid X(s): s \le t) = P(N_1(\lambda(t+h)) - N_1(\lambda t) = 1 \mid X(s): s \le t)$$

$\boxed{\text{Poisson Memoryless Property}}$ $= P(N_1(\lambda(t+h)) - N_1(\lambda t) = 1) = 1\cdot \lambda h + o(h)$

Long Run Behavior OF CTS Time MC.

Want $\lim_{t\to\infty} P_{ij}(t) = \lim_{t\to\infty} P(X(t) = j \mid X(0) = i)$ = long run Prob. of being in state j

Note this limit may or may not exist.

A continuous time MC X is irreducible if for each $i, j \in S$ with $i \ne j$, $P_{ij}(t) > 0$ for all $t > 0$.

Continuous time MC is irreducible iff the discrete-time skeleton is irreducible!

3) Unlike Discrete ~~time~~ MCs,

Don't have to worry about periodicity of cts time MC.

For cts time MCs:

- A state i is <u>recurrent</u> if:

$$P_i\left(\{t \geq 0 : X(t) = i\} \text{ is unbounded} \,\middle|\, X(0) = i\right) = 1.$$

- A state i is <u>transient</u> if:

$$P_i\left(\{t \geq 0 : X(t) = i\} \text{ is unbounded} \,\middle|\, X(0) = i\right) = 0.$$

- Cts time MC is transient/recurrent iff its discrete-time skeleton is transient/recurrent.

- A state i is recurrent: either $Q_{ii} = 0$ or $Q_{ii} < 0$ and $P(T_i < \infty \mid X(0) = i) = 1$, where $T_i = \inf\{t \geq J_1 : X(t) = i\}$, $J_1 = \inf\{t \geq 0 : X(t) \neq X(0)\}$ — the first jump time.

- A state i is transient $\iff$ $Q_{ii} < 0$ & $P(T_i < \infty \mid X(0) = i) < 1$

- A state i is <u>positive recurrent</u> if either $Q_{ii} = 0$ or $E[T_i \mid X(0) = i] < \infty$

(if you have cts time MC that is irreducible with finitely many states, then it is positive recurrent).

<u>Stationary Distributions</u>: a prob. distribution $\pi = (\pi_0, \pi_1, \dots)$ such that $\pi' Q = 0$ (An algebraic analogy to $\pi' P = \pi'$ in discrete time MCs)

Thm: Suppose cts time MC is irreducible, then the MC is positive recurrent iff $\exists$ a stationary distribution π and MC does not explode

In this case (of positive recurrence),

$$\lim_{t\to\infty} P_{ij}(t) = \pi_j \quad \text{for all } i, j \in S.$$

Thm: Suppose cts time MC is irreducible and recurrent and π is a stationary distribution. then $\pi' P(t) = \pi'$ for all $t > 0$.

Idea of proof:

Recall (under nice conditions): $\dfrac{dP(t)}{dt} = Q P(t) \Rightarrow \dfrac{d\pi' P(t)}{dt} = \underbrace{\pi' Q}_{} P(t) = 0$

$\Uparrow$ condition $\pi' Q = 0$.

$\Rightarrow \pi' P(t) = \text{const}$ for all $t > 0$

$= \pi'$ (since $P(0) = I$).

Detailed Balance: A prob. distribution π satisfies detailed balance

if $\pi_i Q_{ij} = \pi_j Q_{ji}$

$$\begin{tikzcd} \underset{\pi_i}{(i)} \arrow[r, bend left, "Q_{ij}"] & \underset{\pi_j}{(j)} \arrow[l, bend left, "Q_{ji}"] \end{tikzcd}$$

for all $j \neq i$.

Check that: π is a stationary distribution if detailed balance holds: sum over all $j \in S$ in $\pi_i Q_{ij} = \pi_j Q_{ji}$.

$$\sum_{j\in S} (\pi_i Q_{ij}) = \sum_{j \in S} (\pi_j Q_{ji})$$

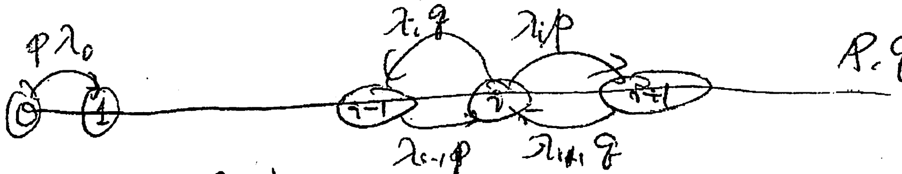
$$\|$$

$$\pi_i \sum_{j\in S} Q_{ij} = \cancel{\pi_i \sum_{j\in S} Q_{ij}}^{\,=0}$$

$\Leftrightarrow (\pi' Q)_i = 0$ for all i.

$\therefore$ Detailed Balance implies: $\pi' Q = 0$.

Example: Birth-Death process:



$p, q > 0$, fixed.

Discrete time skeleton is just a simple random walk with reflection at end 0. Recall: Irreducible, recurrent iff $p \le q$.

Recurrent $\Rightarrow$ nonexplosion of cts time MC.

Seek π prob. distribution that satisfies detailed balance.

$$\pi_i Q_{ij} = \pi_j Q_{ji} \quad \forall j \neq i.$$

Fix $i \geq 0$: $\quad \pi_i Q_{i,i+1} = \pi_{i+1} Q_{i+1,i} \quad \forall i \in S$

$$\Longrightarrow \pi_i \lambda_i p = \pi_{i+1} \lambda_{i+1} q \Rightarrow \pi_{i+1} = \left(\frac{\lambda_i}{\lambda_{i+1}}\right) \frac{p}{q} \pi_i, \quad i = 0, 1, 2 \ldots$$

By Iteration: $\quad \pi_i = \dfrac{\lambda_0}{\lambda_i} \left(\dfrac{p}{q}\right)^i \pi_0, \quad i = 0, 1, 2 \ldots$

If $\displaystyle\sum_{i=1}^{\infty} \frac{1}{\lambda_i} \left(\frac{p}{q}\right)^i < \infty$ then one can choose π_0 so that prob. distribution for π.

<u>Case 1</u>: $\lambda_i = \lambda > 0, \ \forall i$

$\pi_i = \left(\dfrac{p}{q}\right)^i \pi_0$ summable iff $p < q \Rightarrow$ positive recurrent

$\Rightarrow$ Stationary distributions .....

<u>Case 2</u>: $\lambda_i = 2^i, \ \forall i$ (Next time)