MATH 285A: Lecture 13

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Recall from last time:

<u>Q-matrix</u>

$$\begin{pmatrix} \ddots & & \\ & Q_{ii} \;\; Q_{ij} & \\ & & \ddots \end{pmatrix} \leftarrow row\ sum = 0; \quad Q_{ii} = -q(i)$$

<u>(P, q)</u>

$q(i) = -Q_{ii}$, for all $i \in \mathbb{S}$.

$$P_{ij} = \begin{cases} \frac{Q_{ij}}{q(i)}, & i \neq j,\ q(i) \neq 0 \\ 0, & i = j,\ q(i) \neq 0 \\ 0, & i \neq j,\ q(i) = 0 \\ 1, & i = j,\ q(i) = 0 \end{cases}$$

EXAMPLES

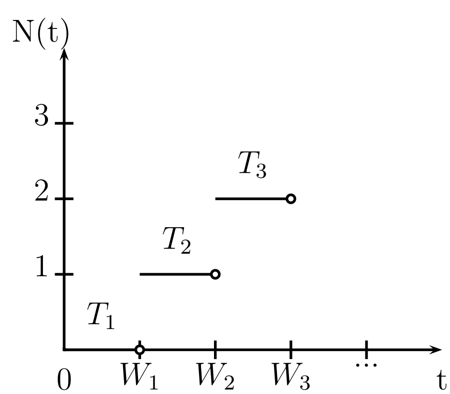
1. POISSON PROCESS - Start from 0;

- Radioactive decay, telephone calls, # hits on website...

- N(t)= # events that have occurred up to time t;

- Interevent times are given by an sequence of i.i.d. exponential random variables.

Typical sample path:

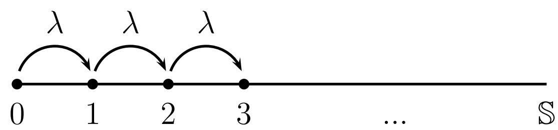


$\{T_i\}_{i=1}^{\infty}$ are i.i.d. exponential with parameter $\lambda > 0$.

$$P(T > t) = e^{-\lambda t}, t \geq 0$$

<u>Continuous time M.C.</u>

$\mathbb{S} = \mathbb{N} = \{0, 1, 2, ...\}$

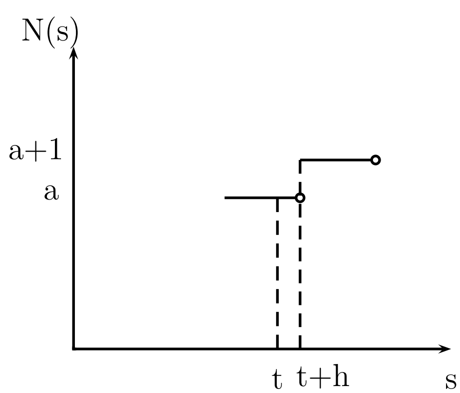


$$Q = \begin{pmatrix} -\lambda & \lambda & 0 & \cdots & 0 \\ 0 & -\lambda & \lambda & 0 & \\ 0 & 0 & -\lambda & \lambda & \\ \vdots & & & \ddots & \ddots \\ 0 & & & & \ddots \end{pmatrix}$$

$q(i) = \lambda, \forall i.$

Non-explosion since $\sup_i q(i) < \infty$.

Theorem. If N_1, N_2 are independent Poisson processes with parameters λ_1 and λ_2, then $N = N_1 + N_2 = \{N_1(t) + N_2(t), t \geq 0\}$ is a Poisson process with parameter $\lambda_1 + \lambda_2$.



Sketch of part of the proof:

$\mathrm{P}(\mathrm{N}(\mathrm{t}+\mathrm{h})-\mathrm{N}(\mathrm{t})=1|\mathrm{N}(\mathrm{s}):\mathrm{s}<\mathrm{t})$ $\quad h$ *small*

$=\mathrm{P}(\mathrm{N}_1(\mathrm{t}+\mathrm{h})-\mathrm{N}_1(\mathrm{t})=1,\mathrm{N}_2(\mathrm{t}+\mathrm{h})-\mathrm{N}_2(\mathrm{t})=0;\text{ or }\mathrm{N}_1(\mathrm{t}+\mathrm{h})-\mathrm{N}_1(\mathrm{t})=0,\mathrm{N}_2(\mathrm{t}+\mathrm{h})-\mathrm{N}_2(\mathrm{t})=1|\mathrm{N}(\mathrm{s}):\mathrm{s}\leq\mathrm{t})$

$=\mathrm{P}(\mathrm{N}_1(\mathrm{t}+\mathrm{h})-\mathrm{N}_1(\mathrm{t})=1,\mathrm{N}_2(\mathrm{t}+\mathrm{h})-\mathrm{N}_2(\mathrm{t})=0)+\mathrm{P}(\mathrm{N}_1(\mathrm{t}+\mathrm{h})-\mathrm{N}_1(\mathrm{t})=0,\mathrm{N}_2(\mathrm{t}+\mathrm{h})-\mathrm{N}_2(\mathrm{t})=1)$

$=\mathrm{P}(\mathrm{N}_1(\mathrm{t}+\mathrm{h})-\mathrm{N}_1(\mathrm{t})=1)\mathrm{P}(\mathrm{N}_2(\mathrm{t}+\mathrm{h})-\mathrm{N}_2(\mathrm{t})=0)+\mathrm{P}(\mathrm{N}_1(\mathrm{t}+\mathrm{h})-\mathrm{N}_1(\mathrm{t})=0)\mathrm{P}(\mathrm{N}_2(\mathrm{t}+\mathrm{h})-\mathrm{N}_2(\mathrm{t})=1)$

$\nwarrow$ (*independenceof* N_1 *and* N_2)

$=(\lambda_1\mathrm{h})(1-\lambda_2\mathrm{h})+(1-\lambda_1\mathrm{h})\lambda_2\mathrm{h}+o(\mathrm{h})$

$=\lambda_1\mathrm{h}+\lambda_2\mathrm{h}+o(\mathrm{h})$

$=(\lambda_1+\lambda_2)\mathrm{h}+o(\mathrm{h})$

$\Rightarrow Q_{i,i+1}=\lambda_1+\lambda_2$

$P(N(t+h)-N(t)\geq 2|N(u):u\leq t)=o(\mathrm{h})$

$\Rightarrow \mathrm{Q}_{\mathrm{ij}}=0,\forall j>i+1;$

$clearly, Q_{ij}=0,\forall j<i;$

$Q_{ii}=-(\lambda_1+\lambda_2).$

Q matrix for N_1+N_2

$$Q=\begin{pmatrix} -(\lambda_1+\lambda_2) & (\lambda_1+\lambda_2) & 0 & \cdots \\ 0 & -(\lambda_1+\lambda_2) & (\lambda_1+\lambda_2) & 0 \\ \vdots & & \ddots & \ddots \\ & & & \end{pmatrix}$$

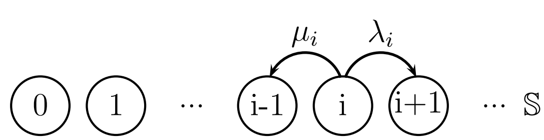
2. BIRTH-DEATH PROCESS

$\mathbb{S}=\mathbb{N}=\{0,1,2,...\}$

-epidemic models ($X_t=\#$ infectives);

-queueing models;

-biochemical reactions...



$$Q = \begin{pmatrix} & & \ddots & & & & \\ 0 & \cdots & \mu_i & -(\lambda_i+\mu_i) & \lambda_i & \cdots & 0 \\ & & & & \ddots & & \end{pmatrix}$$

$$Q_{ij} = \begin{cases} \lambda_i, & if\ j = i+1 \\ \mu_i, & if\ j = i-1, i \geq 1 \\ -(\lambda_i+\mu_i), & if\ j = i \\ 0, & else. \end{cases}$$

Assume that $\mu_0 = 0$ here (for convenience of notation).

Given Markov Chain is in state i, exponential alarm clock goes off at rate $\lambda_i + \mu_i$. When about to jump, go to i+1 with probability $\frac{\lambda_i}{\lambda_i+\mu_i}$ and to i-1 with probability $\frac{\mu_i}{\lambda_i+\mu_i}$.

(i) Pure Birth $\mu_i = 0, \forall i$.

Poisson process: $\lambda_i = \lambda, \forall i$

(ii) Pure Death $\lambda_i = 0, \forall i$

(iii) Yule Process (pure birth)

$\lambda_i = i\lambda, \forall i$

$X_t = \#$ individuals at time t.

Note: $\sup_i(i\lambda) = +\infty$. However, does not explode in finite time as $\sum_i \frac{1}{i\lambda} = +\infty$.

(Use theorem that shows that for pure birth processes, explosion occurs almost surely, if and only if $\sum_i \frac{1}{\lambda_i} < +\infty$.)

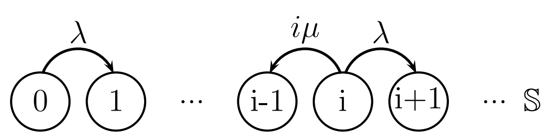
(iv) Simple Biochemical Reaction

One type of molecule A

$\emptyset \xrightarrow{\lambda} A$

$A \xrightarrow{\mu} \emptyset$ (degradation)

$X_t = \#$ molecules of A present at time t.



$\lambda_i = \lambda, \forall i;$

$\mu_i = i\mu, \forall i.$