Math 285A
Stochastic Processes

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Introduction

1.1 Definition

One can think of a stochastic process as a dynamic model for a random phenomenon. A key feature is that the random state of the system can vary with a parameter that is typically thought of as time.

More precisely, a *stochastic process* is a collection of random variables $X \equiv \{X_t : t \in \mathbb{T}\}$. We sometimes write X_t as $X(t)$.

Here $\mathbb{T}$ is the index set and usually this is a set of times at which the stochastic process is observed. Accordingly we shall refer to $\mathbb{T}$ as the time index set. Examples of common index sets are $\mathbb{T} = \{0, 1, 2, \ldots, T\}$ for some finite positive integer T, $\mathbb{T} = \{0, 1, 2, \ldots\} \equiv \mathbb{N}$, $\mathbb{T} = \{\ldots, -2, -1, 0, 1, 2, \ldots\} \equiv \mathbb{Z}$, $\mathbb{T} = [0, T]$ for some finite positive real number T, $\mathbb{T} = [0, \infty) \equiv \mathbb{R}_+$ and $\mathbb{T} = (-\infty, \infty) \equiv \mathbb{R}$. The first three examples are where the index set is discrete (i.e., finite or countably infinite) and the last two are examples where the index set is continuous (in general, we will abuse terminology somewhat and use continuous to mean not discrete). The index can be even more general, e.g., it can be d-dimensional Euclidean space $\mathbb{R}^d$, or a subset thereof, and then X is called a random field.

The random variables $X_t, t \in \mathbb{T}$, take values in a set $\mathbb{S}$ called the *state space.* In specifying the set $\mathbb{S}$ one needs to also give a measurable structure described by a σ-algebra $\mathcal{S}$ of subsets of $\mathbb{S}$. Examples of common state spaces are $\mathbb{S} = \mathbb{N}$, $\mathbb{S} = \mathbb{Z}$, $\mathbb{S} = \mathbb{R}$, and their multidimensional analogues $\mathbb{S} = \mathbb{N}^d$, $\mathbb{S} = \mathbb{Z}^d$ and $\mathbb{S} = \mathbb{R}^d$. The state spaces $\mathbb{N}, \mathbb{Z}, \mathbb{N}^d, \mathbb{Z}^d$ are discrete, being countable, and usually are endowed with the power set (the collection of all possible subsets of $\mathbb{S}$) as the σ-algebra $\mathcal{S}$. The state spaces $\mathbb{R}, \mathbb{R}^d$ are continuous and are usually endowed with their Borel σ-algebras, which are the smallest σ-algebras containing all of the open sets (or even open balls). The state space can be even more general, e.g., it could be a Polish space (a complete separable metric space) endowed with the σ-algebra generated by the open sets. An example of such a space is the set of probability measures on the real line endowed with the topology of weak convergence.

The random variables $\{X_t, t \in \mathbb{T}\}$ are all defined on the same probability space $(\Omega, \mathcal{F}, P)$, where Ω is the sample space, $\mathcal{F}$ is the collection of events (subsets of Ω to which we can assign probabilities) and P is the probability measure on $(\Omega, \mathcal{F})$. The collection $\mathcal{F}$ is a σ-algebra, i.e., it contains the empty set and the whole space Ω, it is closed under taking complements and countable unions. The condition that X_t is a random variable means that the mapping $X_t : \Omega \to \mathbb{S}$ is measurable, i.e., for each set $A \in \mathcal{S}$, $X_t^{-1}(A) \equiv \{\omega \in \Omega : X_t(\omega) \in A\}$ is in $\mathcal{F}$.

Frequently, in accord with common practice, we shall use the term process rather than the longer term stochastic process and we shall often suppress explicit indication of the dependence of $X_t(\omega)$ on $\omega \in \Omega$.

1.2 Classification

It is convenient to classify stochastic processes according to whether their time index sets are discrete or continuous (i.e., not discrete) and as to whether their state spaces are discrete or continuous (i.e., not discrete). Here we mention some examples in each category to provide a guide to the reader. Some terms used here have not yet been defined, but they will be defined when the processes are introduced later in the course.

Markov chains are an important class of discrete state stochastic processes. The term chain signals that the state space is discrete. There are discrete time and continuous time Markov chains. Simple random walk is an example of the former, whereas Poisson processes provide examples of the latter. Renewal processes are more general continuous time, discrete state stochastic processes than Poisson processes and are generally not Markov chains (except in the case when they coincide with Poisson processes).

Time series, frequently used in economics, provide examples of discrete time stochastic processes that may have either discrete or continuous state, depending on the variables being modeled. Examples of discrete time stochastic processes that have continuous state are processes that record successive lifetimes of a component or that record successive arrival times.

Sometimes even though the state space for a stochastic process is discrete, the distance between values in the state space may be so small that it is more convenient to embed the discrete states in a continuous state space and use that instead.

Stochastic processes that are continuous in time and state are often used to model dynamic phenomena that change in a continuous manner and yet are subject to infinitesimal stochastic disturbances. Dynamical systems subject to noise are the most common examples of this. Frequently the driving noise in such equations is modeled by Brownian motion — an important continuous time, continuous state stochastic process. Usually the associated solution of the equation is a Markov process.

Some processes move by jumping at random times (taking a continuum of values) where the jump sizes can also take values in a continuum. Such processes are naturally modeled using a stochastic process with continuous time index and a continuous state space, although the dynamics will typically be governed by an integral equation rather than a differential equation. Lévy processes are examples of continuous time, continuous state processes that typically have some jumps (except when they reduce to Brownian motion) and which can be the source of noise in an integro-differential equation governing the state of a system. Often the solution of such an equation is a Markov process (with jumps).

Gaussian processes have special distributional characteristics. They have continuous state, but can have either discrete or continuous time index. Brownian motion is an example of a continuous time, continuous state Gaussian process, and it is also a Markov process.

Study of processes that have discrete state often involves algebraic or combinatorial manipulations. Processes that are continuous in state and time often involve (continuous) analysis, especially partial differential equations.

1.3 Examples

To illustrate the flexibility in modeling provided by stochastic processes, we give some examples here of stochastic processes arising in applications. Our aim is simply to briefly describe the motivating application and to identify a useful stochastic process for the problem of interest.

Example 1: Molecular Biology.

A segment of DNA consists of a finite sequence of letters, where each entry in the sequence is taken from a finite alphabet $\{A, C, G, T\}$. We are interested in modeling the evolution of a particular segment of DNA with each generation. Describe a stochastic process that will model this time evolution.

Suppose that the segment of DNA is of length m, i.e., it is a finite sequence of length m. Let $\mathbb{A} = \{A, C, G, T\}$ and $\mathbb{S} = \mathbb{A}^m$, the m-fold product of $\mathbb{A}$'s. Endow $\mathbb{A}^m$ with the power set as its σ-algebra. A given segment of DNA is regarded as an element of $\mathbb{A}^m$. Let $\mathbb{T} = \{0, 1, 2, \ldots\}$. For each $n \in \mathbb{T}$, let X_n denote the DNA segment realized in the n^{th} generation, where $n = 0$ corresponds to the original ancestor.

The manner in which the state X_n evolves with time will depend on what modeling assumptions one makes about the forces governing evolution, such as mutation. An example of a question that one might ask is: given an observation of X_N at generation N, what is the most likely ancestor X_0 to have produced this observation?

Example 2: Earth Science.

The temperature is recorded at 20 locations in the Pacific Ocean on a daily basis. Describe a stochastic process that keeps track of these measurements.

Let $\mathbb{T} = \{0, 1, 2, \ldots\}$ and $\mathbb{S} = \mathbb{R}_+^{20}$ endowed with the Borel σ-algebra. Let X_n denote the 20-dimensional vector recording the temperatures at each of the 20 locations on day n, where day 0 is the initial day.

Example 3: Discrete Event Systems.

Stochastic processes that count the number of discrete events that have happened by a given time $t \in \mathbb{T} = \mathbb{R}_+$ arise naturally in various scientific and engineering applications. Here $\mathbb{S} = \mathbb{N}$. For example, in computer science, X_t can be the number of hits on a web site by time t, or in operations research it can be the number of phone calls waiting to be answered by a call center at time t, or in epidemiology it can be the number of people infected by a disease at time t.

Example 4: Simplified Model of Neuron Firing (stochastic Bonhoeffer-Van der Pol oscillator).

A simplified stochastic version of the four-dimensional Hodgkin-Huxley model of neuron firing has state descriptor $X(t) = (X_1(t), X_2(t))$, where $X_1(t)$ denotes the negative of the membrane voltage at time t and $X_2(t)$ is the membrane permeability, and

$$dX_1(t) = c\left(X_1(t) + X_2(t) - \frac{1}{3}(X_1(t))^3 + z\right) dt + \sigma dW(t), \tag{1.1}$$

$$dX_2(t) = -\frac{1}{c}\left(X_1(t) + bX_2(t) - a\right) dt, \tag{1.2}$$

and W is a one-dimensional Brownian motion. Here $z, a, b, c, \sigma > 0$ are constants and $\mathbb{T} = \mathbb{R}_+$ and $\mathbb{S} = \mathbb{R}^2$. (See the web page for examples of the behavior of solutions to these equations. The plots shown illustrate for a given realization ω, the movement of the position $(X_1(t, \omega), X_2(t, \omega))$ as time t evolves.)

1.4 Ways of Viewing and Describing Stochastic Processes

There are several different ways of thinking about a stochastic process $\{X_t, t \in \mathbb{T}\}$ as described below.

(i) For each fixed t, $X_t : \Omega \to \mathbb{S}$ is a random variable.

(ii) For each fixed $\omega \in \Omega$, $t \to X_t(\omega)$ is a function from $\mathbb{T}$ into $\mathbb{S}$. Such a function is called a *sample path* of X. For a continuous time process, often the sample paths have some regularity, e.g., continuity or at least right continuity.

(iii) $X : \mathbb{T} \times \Omega \to \mathbb{S}$ is a function given by $X(t, \omega) = X_t(\omega)$.

The relationship between the random variables $X_t, t \in \mathbb{T}$ is usually an important aspect of a stochastic process. Statistically, one often cares about the finite-dimensional distributions associated with X, i.e., the distributions of the random vectors $(X_{t_1}, \ldots, X_{t_n})$ for each finite set $\{t_1, \ldots, t_n\} \subset \mathbb{T}$. These finite dimensional distributions determine the probability *law* of the process. (This law can be regarded as a probability measure on $\mathbb{S}^{\mathbb{T}}$.) Usually one does not write down the finite dimensional distributions, but rather specifies some rule for determining the law of the process or constructs a new process by transformation of some process whose law is already known. Sometimes, sufficient information for useful analysis of a process is provided by the mean function $m_t = E[X_t]$, $t \in \mathbb{T}$, and covariance function $R(s,t) = E[X_s X_t] - E[X_s]E[X_t]$, $s, t \in \mathbb{T}$, for a process (assuming the expectations involved exist and are finite).

1.5 What to get out of this course

This course aims to provide you with (a) concrete knowledge of some common stochastic processes used in modeling, (b) examples of applications involving stochastic processes, (c) ways of visualizing stochastic processes, and (d) methods for the analysis of stochastic processes.